CHAPTER 4: Relations, Functions, and Matrices

Many of the definitions we give in mathematics are in symbolic form, and students often don't understand that a symbolic expression represents a pattern, which they need to be able to recognize in various guises. I begin to stress this back in Chapter 1; tautologies and derivation rules are patterns that must be recognized even when the letters change, or when the letters themselves become expressions. The same notion occurs in this chapter. When a binary relation is defined by \( x \circ y \iff x \text{ is even} \), I encourage students to translate that to "an ordered pair is related if the first component (whatever its name) is even." A function whose action upon the domain elements is described by the equation \( f(x) = x^2 \) is giving a pattern for what to do with its argument (whatever its name), not what to do with "\( x \)."

There are always some misunderstandings about the properties of binary relations. For example, having some ordered pairs of the form \((x,x)\) does not make the relation reflexive. The definition of transitivity does not imply that \( x, y, \) and \( z \) must be distinct elements. A relation that is not symmetric is not necessarily antisymmetric. And relations can satisfy properties in somewhat trivial ways because of the truth table for the logical connective implication. Indeed, exploring the various properties of binary relations in detail provides a good review of both the implication connective, and the meaning of universal and existential quantifiers. Looking into what it means to fail to be reflexive, for example, introduces the negation of a universally quantified expression, which turns into an existentially quantified negated expression to which DeMorgan's laws can be applied, etc.

"What are the properties of a binary relation [or an equivalence relation]?" becomes my opening question for about a week! I also like to assure students, when we first talk about equivalence classes, that they began using equivalence classes in fourth grade, and then I talk about rational numbers and equivalent fractions (Example 14).

The material in Sections 4.2 and 4.3 is not central and can be omitted without difficulty. Topological sorting (Section 4.2) is explained as the process of turning a partial ordering into a total ordering. Another approach to this task is mentioned in Section 6.4 as an application of depth-first search. Section 4.3, relational databases, provides a nice application of the ideas of an n-ary relation, practical examples of set operations of union, intersection, and difference and of predicate logic notation (in relational calculus).

Students have worked with functions for years, but they still have a surprising amount of trouble with the concepts of one-to-one and onto. I really emphasize how to begin when asked to prove that a function is one-to-one or onto. I make up lots of non-mathematical examples in class and ask whether they are one-to-one and/or onto, such as "S is the set of people in this room, T is the set of shoes in this room, \( f \) associates with each person his or her left shoe."

Another technique that causes confusion is composition of cycles. Students have some difficulty at first in understanding that this is really composition, i.e., they apply the first cycle to some element and then don't know what to do with the rest of the cycles.
Order of magnitude of functions is important in analysis of algorithms. The text describes the order of magnitude of sequential search and binary search, which were analyzed in Chapter 2; order-of-magnitude analyses will be done on the graph algorithms of Chapter 6.

The final section in Chapter 4, on matrices, I leave as a review. Most students have seen this material (although the idea of a Boolean matrix is probably new) and need only look it over to recall the definition of matrix multiplication.

**EXERCISES 4.1**

1. a. \((1, 3), (3, 3)\)  
   b. \((4, 2), (5, 3)\)  
   c. \((5, 0), (2, 2)\)  
   d. \((1, 1), (3, 9)\)

2. a. \((1, -1), (-3, 3)\)  
   b. \((19, 7), (41, 16)\)  
   c. \((-3, -5), (-4, 1/2), (1/2, 1/3)\)  
   d. \(((1, 3), (3, 2))\)

3. a. \((0, 2)\)  
   b. \((0, 1)\)  
   c. \((-3, 0), (5, 0), (-5, 0)\)  
   d. \((-2, -2)\)

4. a. \(x \rho y \leftrightarrow x > -1\)  
   b. \(x \rho y \leftrightarrow -2 \leq y \leq 2\)  
   c. \(x \rho y \leftrightarrow x \leq 2 - y\)  
   d. \(x \rho y \leftrightarrow x^2 + 4y^2 \leq 4\)
*5.  a. many-to-many
    b. many-to-one
    c. one-to-one
    d. one-to-many

6.  a. one-to-one
    b. many-to-one
    c. many-to-many
    d. one-to-many

*7.  a. (2, 6), (3, 17), (0, 0)  b. (2, 12)
    c. none                d. (1, 1), (4, 8)

8.  a. reflexive, antisymmetric
    b. symmetric
    c. symmetric, transitive
    d. reflexive, symmetric, transitive
    e. symmetric, antisymmetric, transitive

9.  a. reflexive, transitive
    b. antisymmetric (because x taller than y and y taller than x is always false, the implication is true), transitive
    c. reflexive, symmetric, transitive
    d. antisymmetric (false antecedent)
    e. antisymmetric, transitive (false antecedent in each case)
    f. symmetric
    g. reflexive, symmetric, transitive
    h. none (not transitive - x ρ y and y ρ x does not imply x ρ x)

10. *a. reflexive, transitive
    *b. reflexive, symmetric, transitive
    *c. symmetric
    d. transitive
    e. reflexive, symmetric, transitive
    f. reflexive, symmetric, transitive
    g. symmetric
    h. reflexive, symmetric, transitive
    i. symmetric
    j. reflexive, antisymmetric, transitive
11. (b); the equivalence classes are
   \[ [0] = \{ \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots \} \]
   \[ [1] = \{ \ldots, -8, -5, -2, 1, 4, 7, 10, \ldots \} \]
   \[ [2] = \{ \ldots, -7, -4, -1, 2, 5, 8, 11, \ldots \} \]
   (e); the equivalence classes are sets consisting of squares with equal length sides
   (f); the equivalence classes are sets consisting of strings with the same number of
   characters
   (h); the equivalence classes are sets consisting of sets with the same number of
   elements

12. For example:
   a. \( S \) = set of all lines in the plane, \( x \rho y \leftrightarrow x \) coincides with \( y \) or \( x \) is perpendicular to \( y \). Then \( \rho \) is reflexive (\( x \) coincides with \( x \)) and symmetric (\( x \) coincides with \( y \leftrightarrow y \) coincides with \( x \) or \( x \perp y \rightarrow y \perp x \)) but not transitive (\( x \perp y \) and \( y \perp z \) only implies \( x \) and \( z \) parallel).

   b. \( S \) = set of integers, \( x \rho y \leftrightarrow x^2 \leq y^2 \). Then \( \rho \) is reflexive (\( x^2 \leq x^2 \)) and transitive (\( x^2 \leq y^2 \) and \( y^2 \leq z^2 \rightarrow x^2 \leq z^2 \)) but not symmetric (\( 2 \rho 3 \) but not \( 3 \rho 2 \)).

   c. \( S \) = set of nonnegative integers, \( x \rho y \leftrightarrow x < y \). Then \( \rho \) is not reflexive (\( x \) is not
   less than \( x \)), not symmetric (\( x < y \nRightarrow y < x \)), but is transitive (\( x < y \) and \( y < z \rightarrow x < z \)).

   d. \( S \) = set of integers, \( x \rho y \leftrightarrow x \leq |y| \). Then \( \rho \) is reflexive (\( x \leq |x| \)), but not symmetric
   (\( -2 \rho 3 \) but not \( 3 \rho -2 \)) and not transitive (\( 3 \rho -4 \) and \( -4 \rho 2 \) but not \( 3 \rho 2 \)).

13. a. yes, yes  
   b. yes, yes  
   c. no, yes  
   d. no, yes

14. a. reflexive closure = \( \rho \) itself
   symmetric closure - add \( (1, 0) \), \( (2, 1) \), \( (4, 2) \), \( (6, 4) \)
   transitive closure - add \( (0, 2) \), \( (1, 4) \), \( (2, 6) \), \( (0, 4) \), \( (0, 6) \), \( (1, 6) \)
   b. reflexive closure - add \( (0, 0) \), \( (1, 1) \), \( (2, 2) \), \( (4, 4) \), \( (6, 6) \)
   symmetric closure = \( \rho \) itself
   transitive closure - add \( (0, 0) \), \( (1, 1) \), \( (2, 2) \), \( (4, 4) \), \( (6, 6) \)
   c. reflexive closure - add \( (4, 4) \), \( (6, 6) \)
   symmetric closure = transitive closure = \( \rho \) itself
   d. \( \rho \) is its own closure with respect to all three properties
   e. reflexive closure - add \( (0, 0) \), \( (1, 1) \), \( (2, 2) \), \( (4, 4) \), \( (6, 6) \)
   symmetric closure = transitive closure = \( \rho \) itself

*15. x \( \rho^* \) y \leftrightarrow one can fly from x to y (perhaps by multiple hops) on Take-Your-Chance
   Airlines
16. a. \( \rho = \{(1, 1)\} \)

b. \( \rho = \{(1, 2), (2, 1), (1, 3)\} \)

c. Assume \( \rho \) is asymmetric and not irreflexive. Then for some \( x \in S \), \( (x, x) \in \rho \). But then \( (x, x) \in \rho \) and \( (x, x) \in \rho \), which contradicts asymmetry.

d. Assume \( \rho \) is irreflexive and transitive but not asymmetric. Then for some \( x, y \in S \), \( (x, y) \in \rho \) and \( (y, x) \in \rho \). By transitivity, \( (x, x) \in \rho \), which contradicts irreflexivity.

e. Assume \( \rho \) is nonempty, symmetric, and transitive. Let \( (x, y) \in \rho \). Then \( (y, x) \in \rho \) by symmetry, and \( (x, x) \in \rho \) by transitivity. Therefore \( (x, x) \in \rho \) for some \( x \in S \) and \( \rho \) is not irreflexive.

17. a. No - if the relation is irreflexive, it is its own irreflexive closure. If the relation is not irreflexive, there must be some \( x \in S \) with \( (x, x) \) in the relation; extending the relation will not remove this pair, so no extension can be irreflexive.

b. No - if the relation is asymmetric, it is its own asymmetric closure. If the relation is not asymmetric, there must be two pairs \( (x, y) \) and \( (y, x) \) in the relation; extending the relation will not remove these pairs, so no extension can be asymmetric.

18. A binary relation is a subset of \( S \times S \). The number of different binary relations is the size of the power set of \( S \times S \). There are \( n^2 \) ordered pairs in \( S \times S \), so the size of the power set is \( 2^{n^2} \).

19. a. Let \( x \in \#A \). Then \( x \rho y \) for all \( y \in A \), so by symmetry, \( y \rho x \) for all \( y \in A \), and \( x \in \#A \). Therefore \( \#A \subseteq \#A \). By a similar argument, \( A# \subseteq \#A \) and \( \#A = A# \).

b. Assume \( A \subseteq B \) and let \( x \in \#B \). Then \( x \rho y \) for all \( y \in B \) and since \( A \subseteq B \), \( x \rho y \) for all \( y \in A \). Thus \( x \in \#A \), and \( \#B \subseteq \#A \). Similarly \( B# \subseteq \#A \).

c. Let \( x \in A \) and let \( z \in \#A \). Then \( z \rho y \) for all \( y \in A \), so in particular, \( z \rho x \). Since \( z \) was arbitrary, \( z \rho x \) holds for all \( z \) in \( \#A \), and therefore \( x \in (\#A)# \) and \( A \subseteq (\#A)# \).

d. Let \( x \in A \) and let \( z \in \#A \). Then \( y \rho z \) for all \( y \in A \), so in particular, \( x \rho z \). Since \( z \) was arbitrary, \( x \rho z \) holds for all \( z \) in \( \#A \), and therefore \( x \in \#(A#) \) and \( A \subseteq \#(A#) \).

20. a. 

b. 

c. 

\[ \begin{array}{lll}
\text{c} & \text{b} & \text{c} \\
\text{b} & \text{a} & \text{d} \\
\text{a} & &
\end{array} \]
21. a. a is minimal and least  
c is maximal and greatest  
b. a and d are minimal  
b, c, and d are maximal  
c. \(\emptyset\) is minimal and least  
\(\{a, c\}\) and \(\{a, b\}\) are maximal

22. Reflexivity: If \(x \in A\), then \(x \in S\), so \((x \ll x)\) because \(\ll \) is a reflexive relation on \(S\).  
Symmetry: if \(x, y \in A\) and \(x \ll y\), then \(x, y \in S\) and \(x \ll y\), so \(y \ll x\) because \(\ll\) is symmetric on \(S\).  
Transitivity: if \(x, y, z \in A\) and \(x \ll y\) and \(y \ll z\), then \(x, y, z \in S\), \(x \ll y\), and \(y \ll z\), so \(x \ll z\) because \(\ll\) is transitive on \(S\).

23. a. No least element; minimal elements of 2, 3, 5, 7; greatest element = maximal element = 210. Totally ordered subsets:
\(\{3, 21, 105, 210\}\), \(\{3, 21, 42, 210\}\), \(\{7, 21, 105, 210\}\), \(\{7, 21, 42, 210\}\).

b. 3 is both least and minimal; there is no greatest element; 72, 108 and 162 are maximal; 6 and 9 are unrelated, as are 72 and 54, 72 and 108, 72 and 162, and 108 and 162.

*24. The two graphs are identical in structure.
25. a. \( \rho = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 5), (1, 5), (2, 4), (4, 5), (2, 5)\} \\
b. \( \rho = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (a, d), (b, e), (c, f)\} \\
c. \( \rho = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 4), (4, 5), (1, 4), (1, 5), (2, 5), (1, 3), (3, 4), (3, 5)\} \)

26. Reflexive: \((s_1, t_1) \in (s_1, t_1)\) because both \(s_1 \in \rho \) \(s_1\) and \(t_1 \in \sigma \) \(t_1\) due to reflexivity of \(\rho\) and \(\sigma\).

Antisymmetric: \((s_1, t_1) \in (s_1, t_1) \Rightarrow s_1 \rho s_2 \) \(s_2\) and \(s_2 \rho s_1, t_1 \sigma t_2\) and \(t_2 \sigma t_1 \Rightarrow s_1 = s_2 \) and \(t_1 = t_2\) due to antisymmetry of \(\rho\) and \(\sigma\) \((s_1, t_1) = (s_2, t_2)\).

Transitive: \((s_1, t_1) \in (s_2, t_2)\) \(s_2, t_2\) and \((s_2, t_2) \in (s_3, t_3)\) \(s_3, t_3\) \(s_1 \rho s_2 \) \(s_2 \rho s_3, t_1 \sigma t_2\) and \(t_2 \sigma t_3 \Rightarrow s_1 \rho s_3 \) \(t_1 \sigma t_3\) due to transitivity of \(\rho\) and \(\sigma\) \((s_1, t_1) \rho (s_3, t_3)\).

27. a. \(\rho^{-1} = \{(2, 1), (3, 2), (3, 5), (5, 4)\} \)

b. If \(\rho\) is reflexive, then \(x \rho x\) for all \(x \in S\), so \(x \rho^{-1} x\) for all \(x \in S\).

c. Let \(x \rho^{-1} y\). Then \(y \rho x\) and, because \(\rho\) is symmetric, \(x \rho y\). Therefore \(y \rho^{-1} x\).

d. Let \(x \rho^{-1} y\) and \(y \rho^{-1} x\). Then \(y \rho x\) and \(x \rho y\) so, because \(\rho\) is antisymmetric, \(x = y\).

e. Let \(x \rho^{-1} y\) and \(y \rho^{-1} z\). Then \(y \rho x\) and \(z \rho y\). By the transitivity of \(\rho\), \(z \rho x\) and therefore \(x \rho^{-1} z\).

f. Let \(\rho\) be irreflexive. Then for all \(x \in S\), \((x, x) \not\in \rho\), so for all \(x \in S\), \((x, x) \not\in \rho^{-1}\).

g. Let \(x \rho^{-1} y\). Then \(y \rho x\) and, by the asymmetric property of \(\rho\), \((x, y) \not\in \rho\) so \((y, x) \not\in \rho^{-1}\).

*28. Assume that \(\rho\) is reflexive and transitive on \(S\). Then for all \(x \in S\), \((x, x) \in \rho\), which means \((x, x) \in \rho^{-1}\), so \((x, x) \in \rho \cap \rho^{-1}\) and \(\rho \cap \rho^{-1}\) is reflexive.

Let \((x, y) \in \rho \cap \rho^{-1}\). Then \((x, y) \in \rho\) and \((x, y) \in \rho^{-1}\), which means \((x, y) \in \rho\) and \((y, x) \in \rho\). This implies \((y, x) \in \rho^{-1}\) and \((y, x) \in \rho\), so \((y, x) \in \rho \cap \rho^{-1}\) and \(\rho \cap \rho^{-1}\) is symmetric.

Let \((x, y) \in \rho \cap \rho^{-1}\) and \((y, z) \in \rho \cap \rho^{-1}\). Then \((x, y) \in \rho\) and \((x, y) \in \rho^{-1}\) and \((y, z) \in \rho\) and \((y, z) \in \rho^{-1}\), so that \((x, y) \in \rho\) and \((y, x) \in \rho\) and \((y, z) \in \rho\) and \((z, y) \in \rho\). Because \(\rho\) is transitive, this means \((x, z) \in \rho\) or \((x, z) \in \rho\) and \((x, z) \in \rho^{-1}\), so \((x, z) \in \rho \cap \rho^{-1}\) and \(\rho \cap \rho^{-1}\) is transitive.

29. a. If \((S, \rho)\) is a partially ordered set, then \(\rho\) is reflexive, antisymmetric, and transitive.

By parts (b), (d), and (e) of Exercise 27, \(\rho^{-1}\) is also reflexive, antisymmetric, and transitive on \(S\), so \((S, \rho^{-1})\) is a partially ordered set.

b. 

[Diagram of a partially ordered set]
c. Let \((x, y) \in \rho^{-1} - X\). Then \((x, y) \in \rho^{-1}\) and \(x \neq y\). Therefore \((y, x) \in \rho\). If \((x, y) \in \rho\), then, because \(\rho\) is antisymmetric, \(x = y\), a contradiction. So \((x, y) \in \rho^{-1}\). This proves that \(\rho^{-1} - X \subseteq \rho'\). Now let \((x, y) \in \rho'\). Then not \(x \rho y\), so by total ordering, \(y \rho x\) and \((x, y) \in \rho^{-1}\). If \(x = y\), then \(x \rho y\) by reflexivity of \(\rho\), a contradiction. Thus \((x, y) \in \rho^{-1} - X\), and \(\rho' \subseteq \rho^{-1} - X\).

30. a. Reflexive: \(X \preceq X\) because \(x_i = x_i\), \(1 \leq i \leq k\).
   Antisymmetric: Let \(X \preceq Y\) and \(Y \preceq X\). If \(X \neq Y\), let \(m + 1\) be the first index
   where \(x_{m+1} \neq y_{m+1}\). Then \(x_{m+1} \prec y_{m+1}\) and \(y_{m+1} \prec x_{m+1}\) →
   \(x_{m+1} = y_{m+1}\), a contradiction.
   Transitive: Let \(X \preceq Y\) and \(Y \preceq Z\). Then \(x_p \preceq y_p\) for some \(p \leq k\) and \(y_q \preceq z_q\) for
   some \(q \leq k\). Let \(m = \min(p, q)\). Then \(x_m \preceq z_m\) and \(X \preceq Z\).
   Total: by "otherwise"
   b. bah < be < boo < bug < bugg

*31. a. when; no; all but the last
   b.

```
                    Old
                   /     \
                 King   was
                /     \
               Cole  merry  soul
               /     \
                a
```
Maximal elements: a, merry, soul

32. The enumeration of \(A^*\) is a, b, ..., z, aa, ab, ac, ..., az, ba, bb, ..., bz, ca, ..., cz, ..., za, ..., zz, aaa, aab, aac, ... (each interval is finite)

*33. a. \([a] = \{a, c\} = [c]\)
   b. \([3] = \{1, 2, 3\}\)
   c. \([-1] = \{-5, -3, -1, 1, 3, 5, \ldots\}\)
   d. \([-3] = \{-13, -8, -3, 2, 7, 12, \ldots\}\)

34. If \(x \equiv y \pmod{n}\) then \(x - y = k_1n\) for some integer \(k_1\), or \(x = k_1n + y\). If \(z \equiv w \pmod{n}\)
then \(z - w = k_2n\) for some integer \(k_2\), or \(z = k_2n + w\).
   *a. \(x + z = (k_1n + y) + (k_2n + w) = y + w + (k_1 + k_2)n\), so \(x + z \equiv (y + w) \pmod{n}\)
where \(k_1 + k_2\) is an integer, and \(x + z \equiv y + w \pmod{n}\)
   b. \(x - z = (k_1n + y) - (k_2n + w) = y - w + (k_1 - k_2)n\), so \(x - z \equiv (y - w) \pmod{n}\)
where \(k_1 - k_2\) is an integer, and \(x - z \equiv y - w \pmod{n}\)
c. \( x^n - y^n = (k_1 n + y)^n - y^n = \left[ \sum_{k=0}^{n} C(s, k)(k_1 n)^{s-k} y^k \right] - y^n = \left[ \sum_{k=0}^{n} C(s, k)(k_1 n)^{s-k} y^k \right] + y^n - y^n = n \sum_{k=0}^{n} C(s, k)k_1^{s-k} n^{s-k} y^k = nk_2 \)

where \( k_2 \) is an integer.

35. If \( x \equiv y \pmod{p} \) then \( x - y = kp \) for some integer \( k \) and \( x^2 - y^2 = (x + y)(x - y) = (x + y)kp \) where \( (x + y)k \) is an integer so \( x^2 \equiv y^2 \pmod{p} \). Similarly, if \( x \equiv -y \pmod{p} \) then \( x + y = kp \) for some integer \( k \) and \( x^2 - y^2 = (x + y)(x - y) = (x - y)kp \) where \( (x - y)k \) is an integer so \( x^2 \equiv y^2 \pmod{p} \).

Conversely, if \( x^2 \equiv y^2 \pmod{p} \), then \( x^2 - y^2 = (x + y)(x - y) = kp \) for some integer \( k \). Therefore \( p \) divides \( (x + y)(x - y) \), but because \( p \) is a prime, either \( p \) divides \( x + y \) or \( p \) divides \( x - y \). If \( p \) divides \( x + y \) then \( x + y \) is an integral multiple of \( p \) and \( x \equiv -y \pmod{p} \). If \( p \) divides \( x - y \) then \( x - y \) is an integral multiple of \( p \) and \( x \equiv y \pmod{p} \).

36. a. \{ (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3) \}

b. \{ (a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d) \}

37. reflexive - the color of \( x \)'s cover is the same as the color of \( x \)'s cover

symmetric - if \( x \)'s cover is the same color as \( y \)'s, then \( y \)'s cover is the same color as \( x \)'s
transitive - if \( x \)'s cover is the same color as \( y \)'s and \( y \)'s cover is the same color as \( z \)'s,

then \( x \)'s cover is the same color as \( z \)'s

The equivalence classes are sets consisting of books with the same colored covers.

*38. reflexive - \( (x, y) \rho (x, y) \) because \( y = y \)
symmetric - if \( (x, y) \rho (z, w) \) then \( y = w \) so \( w = y \) and \( (z, w) \rho (x, y) \)

transitive - if \( (x, y) \rho (z, w) \) and \( (z, w) \rho (s, t) \) then \( y = w \) and \( w = t \) so \( y = t \) and \( (x, y) \rho (s, t) \)

The equivalence classes are sets of ordered pairs with the same second components.

39. reflexive - \( (x, y) \rho (x, y) \) because \( x + y = x + y \)

symmetric - if \( (x, y) \rho (z, w) \) then \( x + y = z + w \) so \( z + w = x + y \) and \( (z, w) \rho (x, y) \)

transitive - if \( (x, y) \rho (z, w) \) and \( (z, w) \rho (s, t) \) then \( x + y = z + w \) and \( z + w = s + t \) so \( x + y = s + t \) and \( (x, y) \rho (s, t) \)

The equivalence classes are sets of ordered pairs whose components add to the same value.

40. reflexive - \( x^2 - x^2 = 0 \), which is even

symmetric - if \( x^2 - y^2 = 2n \) then \( y^2 - x^2 = -2n \), which is even.

transitive - if \( x^2 - y^2 = 2n \) and \( y^2 - z^2 = 2m \), then \( x^2 - z^2 = x^2 - y^2 + y^2 - z^2 = 2n + 2m = 2(n + m) \), which is even.

The equivalence classes are the set of even integers and the set of odd integers.
41. Clearly \( P \iff P \) is a tautology. If \( P \iff Q \) is a tautology, then \( P \) and \( Q \) have the same truth values everywhere, so \( Q \iff P \) is a tautology. If \( P \iff Q \) and \( Q \iff R \) are tautologies, then \( P \), \( Q \), and \( R \) have the same truth values everywhere, and \( P \iff R \) is a tautology. The equivalence classes are sets consisting of wffs with the same truth values everywhere.

42. Reflexive: \( \pi_1 \preceq \pi_1 \) because each block of \( \pi_1 \) is a subset of itself due to reflexivity of set inclusion. Antisymmetry and transitivity also follow from the corresponding set inclusion properties.

43. a. 1 The only way to partition a 1-element set is to use the whole set.
   b. 5 Using a combinatorial argument, the partitions are
   \[
   \begin{align*}
   \text{all in one block} &= 1 \\
   \text{2 in one block, 1 in another} &= \binom{3}{2} = 3 \\
   \text{each element separate} &= 1 \\
   \text{Total} &= 5
   \end{align*}
   \]
   or by directly counting
   \[
   \{a, b, c\} \\
   \{a, b\} \{c\} + \{a, c\} \{b\} + \{b, c\} \{a\} \\
   \{a\} \{b\} \{c\}
   \]
   c. 15 Using a combinatorial argument, the partitions are
   \[
   \begin{align*}
   \text{all in one block} &= 1 \\
   \text{3 in one block, 1 in another} &= \binom{4}{3} = 4 \\
   \text{2 in one block, 2 in another} &= \binom{4}{2}/2 = 3 \\
   \text{2 in one block, 2 in separate blocks} &= \binom{4}{2} = 6 \\
   \text{each element separate} &= 1 \\
   \text{Total} &= 15
   \end{align*}
   \]
   or by directly counting
   \[
   \begin{align*}
   \{a, b, c, d\} \\
   \{a, b, c\} \{d\} + \{a, b, d\} \{c\} + \{a, c, d\} \{b\} + \{b, c, d\} \{a\} \\
   \{a, b\} \{c, d\} + \{a, c\} \{b, d\} + \{a, d\} \{b, c\} \\
   \{a, b\} \{c\} \{d\} + \{a, c\} \{b\} \{d\} + \{a, d\} \{b\} \{c\} + \\
   \{c, d\} \{a\} \{b\} + \{b, d\} \{a\} \{c\} + \{b, c\} \{a\} \{d\} \\
   \{a\} \{b\} \{c\} \{d\}
   \end{align*}
   \]

44. a. Partitions of 3 elements into 2 blocks can only be done with 2 elements in one block and 1 in the other, so the answer is the number of ways to select the 2 elements, or \( \binom{3}{2} = 3 \).
   b. Partitions of 4 elements into 2 blocks can be done with 3 elements in one block and 1 in the other (pick the 3 elements out of 4), or two elements in each block (pick 2 elements out of 4 but this determines the other two elements). The answer is \( \binom{4}{3} + \binom{4}{2}/2 = 7 \).
45. The number of blocks in a partition can range from 1 (the whole set) to \(n\) (a single
element in each block). The result follows by the definition of \(S(n,k)\) and the Addition
Principle.

46. \(S(n, 1) = 1\) because the only partition with 1 block is when the block is the entire set
\(S(n, n) = 1\) because the only partition with \(n\) blocks is where each block consists of a
single element

\[
S(n + 1, k + 1) = S(n, k) + (k + 1)S(n, k + 1)
\]

because if the set without \(x\) is partitioned into \(k\) blocks, which can be done in \(S(n, k)\) ways, these blocks together with \(\{x\}\) form
a partition of the original set with \(k + 1\) blocks. If the set without \(x\) is partitioned into \(k + 1\) blocks, which can be done \(S(n, k + 1)\) ways, then \(x\) can be added to any of these
blocks (which can be done \(k + 1\) ways) to give a partition of the original set with \(k + 1\) blocks; by the Multiplication Principle, this can be done \((k + 1)S(n, k + 1)\) ways.

These are the only two possibilities, so the result follows from the Addition Principle.

*47. a. \(S(3, 2) = S(2, 1) + 2S(2, 2) = 1 + 2 \cdot 1 = 3\)

b. \(S(4, 2) = S(3, 1) + 2S(3, 2) = 1 + 2 \cdot 3 = 7\)

48.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1
\end{array}
\]

*49. \(S(4, 3) = 6\)

50. \(S(5, 3) = 25\)

51. A general term of the sum has the form

\[
C(n - 1, k)p_k
\]

where \(0 \leq k \leq n - 1\). This number counts all the partitions in which \(x\) appears in a
block of size \(n - k\), as follows: For each such block, the complement is a set of size \(k\)
that does not include \(x\). The number of sets of size \(k\) that do not include \(x\) is \(C(n - 1, k)\).
Each such set can be partitioned in \(p_k\) ways. By the Multiplication Principle, there are
\(C(n - 1, k)p_k\) partitions where \(x\) appears in a block of size \(n - k\), and by the Addition
Principle, \(P_n\) is the sum over all block sizes \(n - k\) where \(1 \leq n - k \leq n\) or \(0 \leq k \leq n - 1\).

52. a. \(P_1 = C(0, 0)p_0 = 1 \cdot 1 = 1\)

\(P_2 = C(1, 0)p_0 + C(1, 1)p_1 = 1 \cdot 1 + 1 \cdot 1 = 2\)

\(P_3 = C(2, 0)p_0 + C(2, 1)p_1 + C(2, 2)p_2 = 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 = 5\)

\(P_4 = C(3, 0)p_0 + C(3, 1)p_1 + C(3, 2)p_2 + C(3, 3)p_3 = 1 \cdot 1 + 3 \cdot 1 + 3 \cdot 2 + 1 \cdot 5 = 15\)
b. \( P_1 = S(1, 1) = 1 \)
   \( P_2 = S(2, 1) + S(2, 2) = 1 + 1 = 2 \)
   \( P_3 = S(3, 1) + S(3, 2) + S(3, 3) = 1 + 3 + 1 = 5 \)
   \( P_4 = S(4, 1) + S(4, 2) + S(4, 3) + S(4, 4) = 1 + 7 + 6 + 1 = 15 \)

53. a. 25, 49
   b. (3, 4, 5), (0, 5, 5), (8, 6, 10)
   c. (-4, 4, 2, 0), (-6, 6, 0, -2)

**EXERCISES 4.2**

1. Yes; for example: 1, 2, 3, 8, 4, 5, 6, 7, 9

*2.

\[\text{Diagram as described in text}\]

3.

\[\text{Diagram as described in text}\]

*4. Minimum time-to-completion is 17 time units. Critical path: E, A, D, B, H

5. Minimum time-to-completion is 16 time units. Critical path: 8, 3, 2, 1 or 8, 3, 7, 5, 6

6. For example: G, H, F, D, E, C, A, B

*7. For example: E, A, C, D, G, F, B, H

8. For example: 8, 3, 4, 7, 5, 6, 2, 1

9. The PERT chart is

\[\text{Diagram as described in text}\]

One topological ordering is 6, 9, 1, 7, 8, 11, 2, 3, 5, 10, 4
10. The PERT chart is

One topological ordering is 2, 3, 5, 9, 7, 1, 4, 6, 8

11. The PERT chart is

One topological ordering is 5, 6, 4, 1, 2, 3, 7, 10, 8, 11, 9

EXERCISES 4.3

*1.

The "writes" relation is many-to-many, that is, one author can write many books, and one book can have more than one author.
3.

<table>
<thead>
<tr>
<th>Author</th>
<th>Name</th>
<th>Country</th>
<th>Title</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Book</th>
<th>Title</th>
<th>ISBN</th>
<th>Publisher</th>
<th>Subject</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Writes</th>
<th>Name</th>
<th>Title</th>
<th>ISBN</th>
</tr>
</thead>
</table>

The Author relation requires a composite primary key because one author can write more than one book and one book can have more than one author. The implicit business rule is that one author does not write multiple books with the same title (otherwise the ISBN or some other unique book attribute would need to be an attribute in the author relation.) Tuples in the Book relation are uniquely identified by ISBN. For the Writes relation, both Name and Title are required as a foreign key into the Author relation, and ISBN is required as a foreign key into the Book relation. While a composite primary key is required, it could be either Name and ISBN, or Name and Title because of the implicit business rule.

4.

<table>
<thead>
<tr>
<th>Results1</th>
<th>Name</th>
<th>Country</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bert Kovalsko</td>
<td>U.S.</td>
<td>Baskets for Today</td>
<td></td>
</tr>
<tr>
<td>Jane East</td>
<td>U.S.</td>
<td>Springtime Gardening</td>
<td></td>
</tr>
</tbody>
</table>

5.

<table>
<thead>
<tr>
<th>Results2</th>
<th>Name</th>
<th>Title</th>
<th>ISBN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dorothy King</td>
<td>Autumn Annuals</td>
<td>0-816-88506-0</td>
<td></td>
</tr>
<tr>
<td>Dorothy King</td>
<td>Springtime Gardening</td>
<td>0-816-35421-9</td>
<td></td>
</tr>
</tbody>
</table>

6.

<table>
<thead>
<tr>
<th>Results3</th>
<th>Title</th>
<th>ISBN</th>
<th>Publisher</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early Tang Paintings</td>
<td>0-364-87547-X</td>
<td>Bellman</td>
<td>Art</td>
<td></td>
</tr>
<tr>
<td>Springtime Gardening</td>
<td>0-56-000142-8</td>
<td>Swift-Key</td>
<td>Nature</td>
<td></td>
</tr>
</tbody>
</table>
### Results 4

<table>
<thead>
<tr>
<th>Title</th>
<th>ISBN</th>
<th>Publisher</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baskets for Today</td>
<td>0-816-53705-4</td>
<td>Harding</td>
<td>Art</td>
</tr>
</tbody>
</table>

### Results 5

<table>
<thead>
<tr>
<th>Name</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dorothy King</td>
<td>Springtime Gardening</td>
</tr>
<tr>
<td>Jon Nkoma</td>
<td>Birds of Africa</td>
</tr>
<tr>
<td>Won Lau</td>
<td>Early Tang Paintings</td>
</tr>
<tr>
<td>Bert Kovalsko</td>
<td>Baskets for Today</td>
</tr>
<tr>
<td>Tom Quercos</td>
<td>Mayan Art</td>
</tr>
<tr>
<td>Jimmy Chan</td>
<td>Early Tang Paintings</td>
</tr>
<tr>
<td>Dorothy King</td>
<td>Autumn Annuals</td>
</tr>
<tr>
<td>Jane East</td>
<td>Springtime Gardening</td>
</tr>
</tbody>
</table>

### Results 6

<table>
<thead>
<tr>
<th>Name</th>
<th>Country</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dorothy King</td>
<td>England</td>
</tr>
<tr>
<td>Jon Nkoma</td>
<td>Kenya</td>
</tr>
<tr>
<td>Won Lau</td>
<td>China</td>
</tr>
<tr>
<td>Bert Kovalsko</td>
<td>U.S.</td>
</tr>
<tr>
<td>Tom Quercos</td>
<td>Mexico</td>
</tr>
<tr>
<td>Jimmy Chan</td>
<td>China</td>
</tr>
<tr>
<td>Jane East</td>
<td>U.S.</td>
</tr>
</tbody>
</table>

### Results 7

<table>
<thead>
<tr>
<th>Publisher</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harding</td>
<td>Nature</td>
</tr>
<tr>
<td>Bellman</td>
<td>Art</td>
</tr>
<tr>
<td>Loraine</td>
<td>Nature</td>
</tr>
<tr>
<td>Swift-Key</td>
<td>Nature</td>
</tr>
<tr>
<td>Harding</td>
<td>Art</td>
</tr>
</tbody>
</table>

### Results 8

<table>
<thead>
<tr>
<th>Title</th>
<th>ISBN</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>Springtime Gardening</td>
<td>0-816-35421-9</td>
<td>Nature</td>
</tr>
<tr>
<td>Early Tang Paintings</td>
<td>0-364-87547-X</td>
<td>Art</td>
</tr>
<tr>
<td>Birds of Africa</td>
<td>0-115-01214-1</td>
<td>Nature</td>
</tr>
<tr>
<td>Springtime Gardening</td>
<td>0-56-000142-8</td>
<td>Nature</td>
</tr>
<tr>
<td>Baskets for Today</td>
<td>0-816-53705-4</td>
<td>Art</td>
</tr>
<tr>
<td>Autumn Annuals</td>
<td>0-816-88506-0</td>
<td>Nature</td>
</tr>
</tbody>
</table>
13. A tuple would result in which Dorothy King's name was associated with the ISBN for Springtime Gardening written by Jane East.

14. A tuple would result in which Dorothy King's name was associated with the ISBN for Springtime Gardening written by Jane East.

15. a. project(restrict Author where Country = "U.S.")over Title giving Results11.
   b. SELECT Title FROM Author
      WHERE Author.Country = "U.S."
   c. Range of x is Author
      {x.title|x.Country = "U.S."}
   d. Baskets for Today
      Springtime Gardening

16. a. project(join(restrict Book where Publisher = "Harding") and Writes over ISBN)
     over Name giving Results12.
   b. SELECT Name FROM Writes, Book
      AND Book.Publisher = "Harding"
   c. Range of x is Writes
      Range of y is Book
      {x.Name|exists y(y.Publisher = "Harding" and y.ISBN = x.ISBN)}
   d. Dorothy King
      Bert Kovalesco
17. a. project(join(restrict Book where Subject = "Nature") and Author over Title) over Name giving Results13.
   b. SELECT Name FROM Author, Book
      WHERE Author.Title = Book.Title
      AND Book.Subject = "Nature"
   c. Range of x is Author
      Range of y is Book
      \{x.Name|exists y(y.Subject = "Nature" and y.Title = x.Title}\}
   d. Dorothy King
      Jon Nkoma
      Jane East

18. a. project(join(restrict Author where Country = "U.S.") and Writes over Name and Title) and (restrict Book where Subject = "Art") over ISBN) over Publisher giving Results14
   b. SELECT Publisher FROM Book, Author, Writes
      WHERE Author.Country = "U.S."
      AND Book.Subject = "Art"
      AND Author.Name = Writes.Name
      AND Author.Title = Writes.Title
   c. Range of x is Book
      Range of y is Author
      Range of z is Writes
      \{x.Publisher|x.Subject = "Art" and exists y(y.Country = "U.S.
      and exists z(y.Name = z.Name and y.Title = z.Title and z.ISBN = x.ISBN))}\}
   d. Harding

    Jane East
    (Note that Suzanne Fleur does not satisfy the criteria for the query because
    Author.Country = "U.S." is False, Author.Country = NULL is NULL, and False or
    NULL is NULL, which then gets set to False.)

*20. a. The Cartesian product has cardinality \(p \times q\).
    b. If the common attribute is sorted in each table, then the join can be performed by
doing something similar to a merge sort (see Exercise 13 in Section 2.5) on the
common attribute, which means at most \((p + q)\) rows would need to be examined.

**EXERCISES 4.4**

*1. a. Domain = \{4, 5, 6, 7, 8\} codomain = \{8, 9, 10, 11\} range = \{8, 9, 10\}
   b. 8, 10
   c. 6, 7
   d. no, no
2. (a) is a function, one-to-one but not onto
   (b) is not a function
   (c) is a one-to-one, onto function
   (d) is an onto function but not one-to-one

3. a. \{(0, -1), (1, 1), (2, 3)\}
   b. \{(1, 1), (2, 3), (4, 7), (5, 9)\}
   c. \{(-\sqrt{7}, 2\sqrt{7} - 1), (1.5, 2)\}

*4. a. \(f(A) = \{3, 9, 15\}\)
   b. \(f(A) = \{x | x \in \mathbb{Z} \text{ and } (\exists y)(y \in \mathbb{Z} \text{ and } x = 6y)\}\)

5. a. \(f(\mathbb{N}) = \{0, 1, 4, 9, 16, \ldots\}\)
   b. \(f(\mathbb{Z}) = f(\mathbb{N})\)
   c. \(f(\mathbb{R}) = \{x | x \in \mathbb{R}, x \geq 0\}\)

6. a. not a function from \(S\) to \(T\) (not a subset of \(S \times T\))
   b. function
   c. function; one-to-one and onto
   d. not a function from \(S\) to \(T\) (0 has no associated value)
   e. not a function (two values associated with 0)

7. For part (c), \(f^{-1} : T \rightarrow S, f^{-1} = \{(3, 2), (7, 4), (1, 0), (5, 6)\}\)

8. *a. function
   *b. not a function; undefined at \(x = 0\)
   *c. function; onto
   *d. bijection; \(f^{-1} : \{p, q, r\} \rightarrow \{1, 2, 3\}\) where \(f^{-1} = \{(q, 1), (r, 2), (p, 3)\}\)
   *e. function; one-to-one
   *f. bijection; \(h^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) where \(h^{-1}(x, y) = (y-1, x-1)\)
   g. function
   h. function; onto
   i. not a function (undefined for \(x = -1\), no associated value in \(\mathbb{R}\) for \(x < -1\))
   j. bijection; \(f^{-1} : \mathbb{N} \rightarrow \mathbb{N}\) where \(f^{-1} = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}\)
   (note that the function \(f\) is its own inverse)
   k. not a function (no associated value in \(\mathbb{N}\) for \(x = y = 0, z = 1\))
   l. function; one-to-one

9. \(n\) odd
10. \( f \) performs the following mapping:
\[
\begin{align*}
\emptyset & \rightarrow 00 \\
\{a\} & \rightarrow 10 \\
\{b\} & \rightarrow 01 \\
\{a, b\} & \rightarrow 11
\end{align*}
\]
Thus \( f \) maps each of the four elements in the power set of \( \{a, b\} \) to the four binary strings of length two, and is both one-to-one and onto.

11. \( f \) is neither one-to-one nor onto. \( f(\{a, b\}) = f(\{b, c\}) = 2 \), so \( f \) is not one-to-one. There will be no negative numbers in the range of \( f \), so \( f \) is not onto.

*12. \( f \) is neither one-to-one nor onto. \( f(xyx) = f(yyy) = 3 \), so \( f \) is not one-to-one. For any string \( s \), \( f(s) \geq 0 \); there are no strings in \( A^* \) that map to negative values, so \( f \) is not onto.

13. \( f \) is both one-to-one and onto. If \( f(s_1) = f(s_2) \) then \( s_1 = s_2 \) (just reverse the strings again and you get back where you started), so \( f \) is one-to-one. Given any string \( s \) in \( A^* \), let \( y \) be its reverse. Then \( f(y) = s \), so \( f \) is onto.

14. \( f \) is one-to-one but not onto. If \( f(s_1) = f(s_2) \), then \( xs_1 = xs_2 \), which means that \( s_1 = s_2 \), so \( f \) is one-to-one. Nothing maps to the empty string, so \( f \) is not onto.

15. For example, \( f(x) = 1/x \)

16. For example:
   a. \( f = \{(a, x), (b, x), (c, y), (d, y)\} \)
   b. \( f = \{(a, x), (b, x), (c, y), (d, z)\} \)
   c. no

17. \( x \) is an integer

*18. Let \( k \leq x \leq k + 1 \) where \( k \) is an integer. Then \( \lceil x \rceil = k \). Also, \(-k \geq -x > -k - 1 \) so 
\[
\lfloor -x \rfloor = -k \quad \text{and} \quad -\lfloor -x \rfloor = k.
\]

19. Let \( n = \lceil x \rceil \). Then \( n \) is the smallest integer that is greater than or equal to \( x \), so 
\( n - 1 < x \leq n \). Therefore, adding 1 throughout the inequality, \( n - 1 < x + 1 \leq n + 1 \), and \( n + 1 \) is the smallest integer that is greater than or equal to \( x + 1 \). Therefore \( \lceil x \rceil + 1 = n + 1 \)

20. a. Let \( x = 3.6 \). Then \( \lceil x \rceil = \lceil 3 \rceil = 3 \neq x \).
   b. Let \( x = 4.8 \). Then \( \lfloor 2x \rfloor = \lfloor 9.6 \rfloor = 9 \) but \( 2 \lfloor x \rfloor = 2(4) = 8 \).
   c. Let \( x = 3.6, y = 4.8 \). Then \( \lfloor x \rfloor + \lfloor y \rfloor = \lfloor 3.6 \rfloor + \lfloor 4.8 \rfloor = 3 + 4 = 7 \) but
\( \lfloor x + y \rfloor = \lfloor 8.4 \rfloor = 8 \).
   d. Case 1: \( n \leq x < n + 1/2 \). Then \( 2n \leq 2x < 2n + 1 \) so \( \lfloor 2x \rfloor = 2n \), and 
\( n + 1/2 \leq x + 1/2 < n + 1 \) so \( \lfloor x + 1/2 \rfloor = n \). Therefore \( \lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n = \lfloor 2x \rfloor \).
Case 2: \( n + 1/2 \leq x < n + 1 \). Then \( 2n + 1 \leq 2x < 2n + 2 \) so \( \lfloor 2x \rfloor = 2n + 1 \), and 
\( n + 1 \leq x + 1/2 < n + 1 + 1/2 \) so \( \lfloor x + 1/2 \rfloor = n + 1 \). Therefore \( \lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1 = \lfloor 2x \rfloor \).

*21. If \( 2^k < n < 2^{k+1} \) then \( \log(2^k) < \log n < \log(2^{k+1}) \) or \( k < \log n < k + 1 \) and \( \lceil \log n \rceil = k \), 
\( \lceil \log n \rceil = k + 1 \).

22. If \( 2^k \leq n < 2^{k+1} \) then \( \log(2^k) \leq \log n < \log(2^{k+1}) \) or \( k \leq \log n < k + 1 \) so \( k = \lfloor \log n \rfloor \), 
and \( \lfloor \log n \rfloor + 1 = k + 1 \). Also, \( 2^k < 2^{k+1} \leq n + 1 \leq 2^{k+1} \) so \( \log(2^k) < \log(n + 1) \leq \log(2^{k+1}) \) or \( k < \log(n + 1) \leq k + 1 \) and \( \lceil \log(n + 1) \rceil = k + 1 \).

23. \*a. 9 \*b. 0 \ c. 4 \ d. 2 \ [-7 = (-3) \cdot 3 + 2 \]

24. Let \( x = q_1 n + r_1, 0 \leq r_1 < n \) and \( y = q_2 n + r_2, 0 \leq r_2 < n \), so \( x \mod n = r_1 \) and \( y \mod n = r_2 \).

Also,
\[
x - y = (q_1 n + r_1) - (q_2 n + r_2) = (q_1 - q_2)n + (r_1 - r_2)
\]
with \(-n < r_1 - r_2 < n\).

Then \( x \equiv y \mod n \) if and only if \( x - y = (q_1 - q_2)n \) and \( r_1 - r_2 = 0 \), which is true if and only if \( r_1 = r_2 \) or \( x \mod n = y \mod n \).

25. a. \( (1, 1), (2, 0), (3, 1), (4, 0), (5, 1) \)

b. \( c_{A \cup B}(x) = 1 \iff x \in A \) and \( x \in B \iff c_A(x) = 1 \) and \( c_B(x) = 1 \iff c_A(x) \cdot c_B(x) = 1 \)

c. If \( c_A(x) = 1 \), then \( x \in A' \) and \( x \not\in A \), so \( c_A(x) = 0 = 1 - c_A(x) \)

If \( c_A(x) = 0 \), then \( x \not\in A' \) and \( x \in A \) so \( c_A(x) = 1 = 1 - c_A(x) \)

d. No. Let \( S = \{1, 2, 3\} \), \( A = \{1, 2\} \), \( B = \{2, 3\} \). Then \( c_{A \cup B}(2) = 1 \) but 
\( c_A(2) + c_B(2) = 1 + 1 \).

26. a. \( 2x \) \hspace{1cm} b. \( 2^x \) \hspace{1cm} c. \( 2^{16} \)

27. \( g \circ f = \{(1,6), (2,7), (3,9), (4,9)\} \)

*28. a. \( (g \circ f)(5) = g(f(5)) = g(6) = 18 \)

b. \( (f \circ g)(5) = f(g(5)) = f(15) = 16 \)

c. \( (g \circ f)(x) = g(f(x)) = g(x + 1) = 3(x + 1) = 3x + 3 \)

d. \( (f \circ g)(x) = f(g(x)) = f(3x) = 3x + 1 \)

e. \( (f \circ f)(x) = f(f(x)) = f(x + 1) = (x + 1) + 1 = x + 2 \)

f. \( (g \circ g)(x) = g(g(x)) = g(3x) = 3(3x) = 9x \)

29. a. 25 \hspace{1cm} b. 1470 (the index doesn't count) \hspace{1cm} c. 9
30. a. \( g \circ f = 12x^3 \)  \( \quad f \circ g = 48x^3 \)
   b. \( g \circ f = x^2 - 2x + 1 \)  \( \quad f \circ g = (4x^2 - 1)/2 \)
   c. \( g \circ f = \lfloor x \rfloor \)  \( \quad f \circ g = \lfloor x \rfloor \)

31. a. If \( f(s_1) = f(s_2) \) then \( g(f(s_1)) = g(f(s_2)) \) so \( (g \circ f)(s_1) = (g \circ f)(s_2) \). Because \( g \circ f \) is one-to-one, \( s_1 = s_2 \) and therefore \( f \) is one-to-one.

   b. For \( u \in U \), there exists \( s \in S \) such that \( (g \circ f)(s) = u \), because \( g \circ f \) is onto. Thus \( g(f(s)) = u \) and \( f(s) \) is a member of \( T \) that is a preimage of \( u \) under \( g \), and \( g \) is onto.

   c. Let \( S = \{1, 2, 3\} \), \( T = \{1, 2, 3, 4\} \), \( U = \{1, 2, 3\} \),
   \[ f = \{(1, 1), (2, 2), (3, 3)\}, \quad g = \{(1, 1), (2, 2), (3, 3), (4, 3)\} \]. Then \( f: S \to T \), \( g: T \to U \),
   \( g \) is not one-to-one but \( g \circ f = \{(1, 1), (2, 2), (3, 3)\} \) is one-to-one.

   d. same example as for (c)

*32. a. \( f^{-1}(x) = x/2 \)
   b. \( f^{-1}(x) = \sqrt[3]{x} \)
   c. \( f^{-1}(x) = 3x - 4 \)

33. a. Assume \( f \) has a left inverse \( g \), and that \( f(s_1) = f(s_2) \). Then \( g(f(s_1)) = g(f(s_2)) \) or
   \( (g \circ f)(s_1) = (g \circ f)(s_2) \) and \( i_S(s_1) = i_S(s_2) \), thus \( s_1 = s_2 \) and \( f \) is one-to-one. Now
   let \( f: S \to T \) with \( f \) one-to-one. We want to define \( g: T \to S \). For \( t \in T \) with
   \( t \in f(S) \), define \( g(t) \) to be the unique preimage of \( t \) under \( f \). For \( t \in T \) with \( t \in f(S) \),
   let \( g(t) \) be any fixed element of \( S \). Then \( g: T \to S \) and for \( s \in S \), \( (g \circ f)(s) = g(f(s)) \)
   \( = g(t) = s \), so \( g \circ f = i_S \).

   b. Assume \( f \) has a right inverse \( g \), and let \( t \in T \). Then \( t = i_T(t) = (f \circ g)(t) = f(g(t)) \);
   \( g(t) \in S \), so \( t \in f(S) \) and \( f \) is onto. Now let \( f: S \to T \) with \( f \) onto. Then every \( t \in T \)
   has at least one preimage in \( S \) under \( f \). Define \( g: T \to S \) by \( g(t) = a \) a fixed preimage
   \( s \) of \( t \). Then \( (f \circ g)(t) = f(g(t)) = f(s) = t \), so \( f \circ g = i_T \).

   c. For example:
   \[
   g_1(x) = \begin{cases} 
   x/3 & \text{for } x = 3k \quad k \text{ an integer} \\
   0 & \text{for } x \neq 3k 
   \end{cases}
   \]
   \[
   g_2(x) = \begin{cases} 
   x/3 & \text{for } x = 3k \quad k \text{ an integer} \\
   1 & \text{for } x \neq 3k 
   \end{cases}
   \]

   d. \( g_1(x) = 2x \)
   \( g_2(x) = 2x - 1 \)
34. $f^{-1} : T \to S$, $g^{-1} : U \to T$, so $f^{-1} \circ g^{-1} : U \to S$. For $s \in S$, let $f(s) = t$ and $g(t) = u$. Then $(f^{-1} \circ g^{-1}) \circ (g \circ f)(s) = f^{-1}(g^{-1}(u)) = f^{-1}(t) = s$. Also for $u \in U$, $(g \circ f) \circ (f^{-1} \circ g^{-1})(u) = g(f(s)) = g(t) = u$. Then $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathbf{i}_S$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathbf{i}_U$, so $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

*35. a. $(1, 3, 5, 2)$
b. $(1, 4, 3, 2, 5)$

36. a. $
\begin{pmatrix}
  a & b & c & d \\
  c & b & d & a
\end{pmatrix}$
b. $
\begin{pmatrix}
  a & b & c & d \\
  b & d & a & c
\end{pmatrix}$
c. $
\begin{pmatrix}
  a & b & c & d \\
  d & a & c & b
\end{pmatrix}$
d. $
\begin{pmatrix}
  a & b & c & d \\
  c & d & b & a
\end{pmatrix}$

37. Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have domain and codomain $A$. For $x \in A$, $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x)$.

*38. a. $(1, 2, 5, 3, 4)$
b. $(1, 7, 8) \circ (2, 4, 6)$
c. $(1, 5, 2, 4) \circ (3, 6)$
d. $(2, 3) \circ (4, 8) \circ (5, 7)$

39. a. $(1, 6, 4, 8, 3, 5, 2)$
b. $(1, 3) \circ (2, 4) \circ (5, 13, 6)$
c. $(1, 5, 4, 3, 2)$

40. For example, $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$.

41. a. $
\begin{pmatrix}
  1 & 2 & 3 & 4 \\
  2 & 1 & 4 & 3
\end{pmatrix}$
b. For example, $
\begin{pmatrix}
  1 & 2 & 3 & 4 \\
  3 & 1 & 2 & 4
\end{pmatrix}$

*42. a. $3^4$
b. $36$

43. a. $4^3$
b. $4!$

44. a. for $|S| = 2$, $2! = 2$ and $2^2 - C(2, 1)(1)^2 = 4 - 2 = 2$
for $|S| = 3$, $3! = 6$ and $3^3 - C(3, 1)(2)^3 + C(3, 2)(1)^3 = 27 - 3 \cdot 8 + 3 = 6$
for $|S| = 4$, $4! = 24$ and $4^4 - C(4, 1)(3)^4 + C(4, 2)(2)^4 - C(4, 3)(1)^4$
$= 256 - 4 \cdot 81 + 6 \cdot 16 - 4 = 24$
b. Assume \( f \) is onto. If two distinct elements of \( S \) map to one element of \( S \), then \( n - 2 \) elements are left to map onto \( n - 1 \) elements, which cannot be done. Therefore \( f \) is one-to-one. Now assume \( f \) is one-to-one. Then the \( n \) elements of \( S \) map to \( n \) distinct elements of \( S \); thus every element of \( S \) is in the range of \( f \), and \( f \) is onto.

c. For example, \( S = \mathbb{N} \), \( f: \mathbb{N} \rightarrow \mathbb{N} \) given by \( f(x) = 2x \).

\[
\begin{align*}
\{ f(0) &= 0 \\
f(x) &= x - 1, \ x \geq 1 
\end{align*}
\]

d. For example, \( S = \mathbb{N} \), \( f: \mathbb{N} \rightarrow \mathbb{N} \) given by

\[
\begin{align*}
\{ f(0) &= 0 \\
f(x) &= x - 1, \ x \geq 1 
\end{align*}
\]

*45. a. \( n^n \)

b. \( n! \)

c. \( n! \)

d. \( n! \)

e. \( n\left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] = n!\left[ \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] < n!\left[ \frac{1}{2!} \right] = n! \cdot \frac{1}{2} < n! \)

f. Number of derangements \(< n! < n^n \). The total number of functions, with no restrictions, is the maximum. Only some of these functions are one-to-one and onto, but this is the definition of a permutation as well. Not all permutations are derangements, so the number of derangements is smaller still.

46. a. This is the number of onto functions from a set of 5 elements to a set of 3, which is 150.

b. If Maria does additional tasks, then the mapping from the test plan development to Maria is already determined, leaving the remaining 4 tasks to be assigned to 3 workers with each worker getting at least one task. This number is 36. If Maria does no additional task, then the mapping from the test plan development to Maria is already determined, leaving the remaining 4 tasks to be assigned to 2 workers with each worker getting at least one task. This number is 14. By the Addition Principle, the total number of outcomes is 36 + 14 = 50.

47. For a given onto function from a set with \( m \) elements to a set with \( n \) elements, any permutation of the \( n \) images would give a different onto function but would determine the same partition of \( m \) objects into \( n \) blocks. Hence dividing the number of onto functions by \( n! \), the number of image permutations, will give \( S(m, n) \).

48. 24, 9

\[
\begin{pmatrix}
a & b & c & d \\
b & c & d & a \\
c & d & b & a \\
d & a & b & c \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a & b & c & d \\
b & a & c & d \\
c & a & d & b \\
d & c & a & b \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a & b & c & d \\
b & a & d & c \\
c & d & a & b \\
d & c & b & a \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a & b & c & d \\
b & d & a & c \\
c & a & b & d \\
d & b & c & a \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a & b & c & d \\
b & c & d & a \\
c & d & b & a \\
d & b & c & a \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a & b & c & d \\
b & d & a & c \\
c & a & b & d \\
d & b & c & a \\
\end{pmatrix}
\]

*49. This is the number of derangements of 7 items, which is 1854.

50. a. \( e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \)
b. \[ e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \]

c. \[ e^{-1} \approx 0.36788 \]

d. The expression in brackets in Equation (4) is the sum of the first \( n+1 \) terms of the series representation for \( e^{-1} \). Because the terms of the series approach zero as \( n \) gets larger, \( e^{-1} \) is close to the value of this expression for large \( n \). Therefore the expression \( n!e^{-1} \) is a good approximation to (4).

e. \[ 7!e^{-1} \approx 1854.1 \]

f. \[ 10!e^{-1} \approx 1334961 \]

51. a. The values are stored in locations 6, 14, 1, 7, 8, 2, 16, 9, 0.

b. 58 hashes to location 7, which contains another element (40), so, following the collision resolution scheme under which 58 would have stored, search the next table position, 8, which contains 24, then search the next table position, 9, which contains 58. 41 also hashes to location 7 in the table; proceeding as before, locations 8 and 9 are also checked, and do not contain 41. The next location to check is 10, which is empty. Therefore 41 is not in the table.

c. If, say, item 24, stored at location 8, is deleted, then in searching for 58, we would check location 7 and then location 8. Finding location 8 to be empty, we would conclude incorrectly that 58 is not in the table.

*52. a. 3

b. X

53. a. We must have \( 11 \cdot d \mod 8 = 1 \), with \( 0 < d < 8 \), so \( d = 3 \) (\( 11 \cdot 3 = 33 \mod 8 = 1 \)).

b. The code for 3 is \( 3^{11} \mod 15 \). Doing successive reductions mod 15, \( 3^{11} \rightarrow 3^{4} \cdot 3^{3} \)
\[ \rightarrow 81 \cdot 81 \cdot 27 \rightarrow 6 \cdot 6 \cdot 12 \rightarrow 6 \cdot 72 \rightarrow 6 \cdot 12 \rightarrow 72 \rightarrow 12. \]

c. To decode, compute \( (12)^{3} \mod 15 \). Doing successive reductions mod 15,
\[ 12^{2} \rightarrow (12^{2}) \cdot 12 \rightarrow 144 \cdot 12 \rightarrow 9 \cdot 12 \rightarrow 108 \rightarrow 3. \]

54. Reflexive: \( S \cap S \) by the identify function.

Symmetric: If \( S \cap T \) and \( f \) is a bijection from \( S \) to \( T \), then \( f^{-1} \): \( T \rightarrow S \) and \( f^{-1} \) is a bijection, so \( T \cap S \).

Transitive: if \( S \cap T \) and \( T \cap U \), \( f: S \rightarrow T \), \( g: T \rightarrow U \), \( f \) and \( g \) bijections, then \( g \circ f: S \rightarrow U \) and \( g \circ f \) is a bijection, so \( S \cap U \).


56. a. Let \( t \in f(A \cap B) \). Then \( t = f(s) \) for some \( s \in A \cap B \). Thus \( t \in f(A) \) and \( t \in f(B) \), so \( t \in f(A) \cap f(B) \).

b. Assume \( f \) is one-to-one. By (a), \( f(A \cap B) \subseteq f(A) \cap f(B) \). Let \( t \in f(A) \cap f(B) \). Then \( t = f(s_{1}) \) for some \( s_{1} \in A \) and \( t = f(s_{2}) \) for some \( s_{2} \in B \). Because \( f \) is
one-to-one, \( s_1 = s_2 \) and \( s_1 \in A \cap B \), so \( t \in f(A \cap B) \). Thus \( f(A) \cap f(B) \subseteq f(A \cap B) \) and the two sets are equal.

Now assume \( f(A \cap B) = f(A) \cap f(B) \) for all subsets \( A \) and \( B \) of \( S \), and let \( s_1, s_2 \in S \) such that \( f(s_1) = f(s_2) \). Let \( t = f(s_1) \) and let \( A = \{s_1\} \), \( B = \{s_2\} \). Then \( t \in f(A) \cap f(B) \), or \( t \in f(A \cap B) \). Therefore \( A \cap B \neq \emptyset \), and \( s_1 = s_2 \), so \( f \) is one-to-one.

*57. a. \( \{m, n, o, p\} \)
   b. \( \{n, o, p\}; \{o\} \)

58. a. For \( x \in S \), \( f(x) = f(x) \), so \( x \rho x \) and \( \rho \) is reflexive. For \( x, y \in S \), if \( x \rho y \) then \( f(x) = f(y) \) and \( f(y) = f(x) \) so \( y \rho x \) and \( \rho \) is symmetric. For \( x, y, z \in S \), if \( x \rho y \) and \( y \rho z \) then \( f(x) = f(y) \) and \( f(y) = f(z) \) so \( f(x) = f(z) \), and \( x \rho z \), so \( \rho \) is transitive.
   b. \( [4] = \{4, -4\} \)

59. This algorithm does the maximum amount of work when the wff is a tautology, because it must examine every row of the truth table to see that each gives a true result for the wff. If the wff has \( n \) statement letters, there are \( 2^n \) rows to the truth table.

*60. For example, \( n_0 = c_2 = 1 \), \( c_1 = 1/34 \)

61. For example, \( n_0 = 2 \), \( c_1 = 1 \), \( c_2 = 6 \)

62. \( \log(x^2 + 3) = \Theta(\log(x^2)) \) (ignoring low-order terms) = \( \Theta(2 \log x) \) (property of logarithms) = \( \Theta(\log x) \) (ignoring constant coefficient)

63. Yes. For example, in Exercise 60, we could use the constants \( n_0 = 1 \), \( c_1 = 1/34 \), \( c_2 = 1/10 \). Then the envelope would be entirely below \( g(x) \), but it still follows the general "shape" of \( g(x) \).

64. Let \( f_1 = \Theta(g_1) \) and \( f_2 = \Theta(g_2) \). Then there exist positive constants \( n_0, n_1, c_1, c_2, d_1, d_2 \) with

\[
  c_1 g_1(x) \leq f_1(x) \leq c_2 g_1(x) \quad \text{for} \ x \geq n_0
\]

and

\[
  d_1 g_2(x) \leq f_2(x) \leq d_2 g_2(x) \quad \text{for} \ x \geq n_1
\]

Then for all \( x \geq \max(n_0, n_1) \),

\[
  \min(c_1, d_1) \max(g_1(x), g_2(x)) \leq c_1 g_1(x) + d_1 g_2(x) \leq f_1(x) + f_2(x) \leq c_2 g_1(x) + d_2 g_2(x) \leq \max(c_1 g_1(x), g_2(x)) + \max(d_1 g_2(x), g_2(x)) = (c_2 + d_2) \max(g_1(x), g_2(x))
\]

and \( f_1 + f_2 = \Theta(\max(g_1, g_2)) \).
*65. \( \lim_{x \to \infty} \frac{x}{17x + 1} = \lim_{x \to \infty} \frac{1}{17} = \frac{1}{17} \)

66. \( \lim_{x \to \infty} \frac{3x^3 - 7x}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{9x^2 - 7}{\frac{3}{x^2}} = \frac{18x}{3} = 6 \)

*67. \( \lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{2x} = 0 \)

68. \( \lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{1}{x \log e} = 0 \)

69. \( \lim_{x \to \infty} \frac{(\ln x)^2}{x^{0.5}} = 2 \lim_{x \to \infty} \frac{\ln x}{x^{0.5}} = 2 \lim_{x \to \infty} \frac{\ln x}{x^{0.5}} = \lim_{x \to \infty} \frac{1}{x^{0.5}} \cdot \frac{4 \ln x}{x} = \lim_{x \to \infty} \frac{1}{x^{0.5}} = 0 \)

70. \([200 \log x] = [41 \ln x^2] < [\sqrt[4]{x}] < [420 x] < [17 x \log x] < [10x^2 - 3x + 5] < [2^x - x^2] \)

71. \([\log x] = [\ln x] < [(\log x)2] < [\sqrt{x}] < [x] < [x \log x] < [x^3] = [2x^3 + x] = [x^3 + \log x] < [e^x] \)

EXERCISES 4.5

*1. 2, -4

2. \( x = 2, y = 4 \)

*3. \( x = 1, y = 3, z = -2, w = 4 \)

4. \( u = 1, v = -3, w = 7 \)

5. *a. \( \begin{bmatrix} 6 & -5 \\ 0 & 3 \\ 5 & 3 \end{bmatrix} \) b. \( \begin{bmatrix} -2 & 7 \\ -2 & -3 \\ 1 & 5 \end{bmatrix} \) c. \( \begin{bmatrix} 12 & 3 & 6 \\ 18 & -3 & 15 \\ 3 & 9 & 6 \end{bmatrix} \)

d. \( \begin{bmatrix} -4 & -8 \\ -12 & 2 \end{bmatrix} \) *e. \( \begin{bmatrix} 14 & -17 \\ 2 & 9 \\ 9 & 1 \end{bmatrix} \)

f. not possible
g. \[
\begin{bmatrix}
18 & -15 \\
0 & 9 \\
15 & 9
\end{bmatrix}
\]
h. \[
\begin{bmatrix}
-12 & -24 \\
-36 & 6
\end{bmatrix}
\]
i. \[
\begin{bmatrix}
21 & -23 \\
33 & -44 \\
11 & 1
\end{bmatrix}
\]
j. \[
\begin{bmatrix}
-28 & 22 \\
20 & 1 \\
-2 & 9
\end{bmatrix}
\]
k. \[
\begin{bmatrix}
10 & 7 \\
-2 & 4 \\
30 & 8
\end{bmatrix}
\]
l. not possible

*m. \[
\begin{bmatrix}
28 & 4 \\
6 & 25
\end{bmatrix}
\]
n. \[
\begin{bmatrix}
17 & 6 \\
29 & 29 \\
7 & 8
\end{bmatrix}
\]

6. a. \[
A \cdot B = \begin{bmatrix}
10 & 4 \\
18 & -3
\end{bmatrix}
\]
B \cdot A = \begin{bmatrix}
14 & 1 \\
4 & -7
\end{bmatrix}

b. \[
A(B \cdot C) = \begin{bmatrix}
3 & -1 \\
2 & 5
\end{bmatrix}
\begin{bmatrix}
26 & -22 \\
10 & -8
\end{bmatrix}
= \begin{bmatrix}
64 & -58 \\
102 & -84
\end{bmatrix}
\]
\[(A \cdot B)C = \begin{bmatrix}
10 & 4 \\
18 & -3
\end{bmatrix}
\begin{bmatrix}
6 & -5 \\
2 & -2
\end{bmatrix}
= \begin{bmatrix}
68 & -58 \\
102 & -84
\end{bmatrix}
\]

*c. \[
A(B + C) = \begin{bmatrix}
3 & -1 \\
2 & 5
\end{bmatrix}
\begin{bmatrix}
10 & -4 \\
4 & 3
\end{bmatrix}
= \begin{bmatrix}
26 & -9 \\
40 & -23
\end{bmatrix}
\]
\[A \cdot B + A \cdot C = \begin{bmatrix}
10 & 4 \\
18 & -3
\end{bmatrix}
+ \begin{bmatrix}
16 & -13 \\
22 & -20
\end{bmatrix}
= \begin{bmatrix}
26 & -9 \\
40 & -23
\end{bmatrix}
\]

\[\begin{bmatrix}
7 & 0 \\
4 & 2
\end{bmatrix}
\begin{bmatrix}
6 & -5 \\
2 & -2
\end{bmatrix}
= \begin{bmatrix}
42 & -35 \\
32 & -28
\end{bmatrix}
\]
\[A \cdot C + B \cdot C = \begin{bmatrix}
16 & -13 \\
22 & -20
\end{bmatrix}
+ \begin{bmatrix}
26 & -22 \\
10 & -8
\end{bmatrix}
= \begin{bmatrix}
42 & -35 \\
32 & -28
\end{bmatrix}
\]

7. \(x = 3, \ y = 4\)

8. a. \(I \cdot A = A\) for any \(n \times n\) matrix \(A\), in particular, if \(A = I\), then \(I^2 = I\cdot I = I\)

b. \(I^1 = I\). Assume \(I^k = I\). Then \(I^{k+1} = I^k \cdot I = I \cdot I = I\)

9. a. Assume that row \(i\) of \(A\) is all 0s. Then for any \(j\), the element in row \(i\), column \(j\) of \(A \cdot B\) is given by \(\sum_{k=1}^{n} a_{ik}b_{kj}\). This sum is 0 because \(a_{ik} = 0\) for all values of \(k\).

b. Assume that column \(j\) of \(B\) is all 0s. Then for any \(i\), the element in row \(i\), column \(j\) of \(A \cdot B\) is given by \(\sum_{k=1}^{n} a_{ik}b_{kj}\). This sum is 0 because \(b_{kj} = 0\) for all values of \(k\).
10. a. Let $C = A + B$. Then $c_{ij} = a_{ij} + b_{ij}$. If $i \neq j$ then $a_{ij} = b_{ij} = 0$, so $c_{ij} = 0$.

b. Element $i$, $j$ in $rA$ is given by $r(a_{ij})$. If $a_{ij} = 0$, then $ra_{ij} = 0$, hence $rA$ will be diagonal.

c. Let $C = A \cdot B$. Then $c_{ij}$ is given by $\sum_{k=1}^{n} a_{ik} b_{kj}$. Assume that $i \neq j$. Then for any value of $k$, $1 \leq k \leq n$, $k$ cannot equal both $i$ and $j$, so that either $a_{ik}$ or $b_{kj}$ (or both) is 0. Therefore each term in the summation is 0 and $c_{ij} = 0$.

11.*a. \[
\begin{pmatrix}
1 & 3 \\
2 & 4
\end{pmatrix}
\begin{pmatrix}
-1/2 & 3/4 \\
1/2 & -1/4
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
-1/2 & 3/4 \\
1/2 & -1/4
\end{pmatrix}
\begin{pmatrix}
1 & 3
\end{pmatrix}
\]

*b. For \[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
= \begin{pmatrix}
1 & 0
\end{pmatrix}
\]

\[
b_{11} + 2b_{21} = 1 \\
2b_{11} + 4b_{21} = 0
\]

which is an inconsistent system of equations with no solution.

c. \[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
= \begin{pmatrix}
1 & 0
\end{pmatrix}
\]

implies

\[
a_{11} b_{11} + a_{12} b_{21} = 1 \quad a_{11} b_{12} + a_{12} b_{22} = 0 \\
a_{21} b_{11} + a_{22} b_{21} = 0 \quad a_{21} b_{12} + a_{22} b_{22} = 1
\]

Solving these systems of equations gives

\[
b_{11} = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} \\
b_{12} = \frac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \\
b_{21} = \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \\
b_{22} = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}}
\]

These values can all be found if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

12. $(rA)(1/r)A^{-1} = r(1/r)(A \cdot A^{-1}) = I I = I$

$(1/r)A^{-1}(rA) = (1/r)(r)(A^{-1} \cdot A) = I I = I$

13. If $A$ is invertible, then $A^{-1}$ exists, and

\[
A^{-1}(A \cdot B) = A^{-1}(A \cdot C) \\
(A^{-1} \cdot A)B = (A^{-1} \cdot A)C \\
IB = IC \\
B = C
\]
14. a. Here are the steps:

\[
A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Multiply row 1 by -2 and add to row 2:

\[
\begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}
\]

Multiply row 2 by -1/4:

\[
\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & -1/4 \end{bmatrix}
\]

Multiply row 2 by -3 and add to row 1:

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 3/4 \\ 1/2 & -1/4 \end{bmatrix} = A^{-1}
\]

b. Here are the steps:

\[
\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Multiply row 1 by -1:

\[
\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Multiply row 1 by -2 and add to row 2:

\[
\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
Multiply row 1 by -4 and add to row 3:

\[
\begin{bmatrix}
1 & -2 & 3 \\
0 & 5 & -6 \\
0 & 6 & -7
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 \\
2 & 1 & 0 \\
4 & 0 & 1
\end{bmatrix}
\]

Multiply row 2 by 1/5:

\[
\begin{bmatrix}
1 & -2 & 3 \\
0 & 1 & -6/5 \\
0 & 6 & -7
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 \\
2/5 & 1/5 & 0 \\
4 & 0 & 1
\end{bmatrix}
\]

Multiply row 2 by 2 and add to row 1:

\[
\begin{bmatrix}
1 & 0 & 3/5 \\
0 & 1 & -6/5 \\
0 & 6 & -7
\end{bmatrix}
\begin{bmatrix}
-1/5 & 2/5 & 0 \\
2/5 & 1/5 & 0 \\
4 & 0 & 1
\end{bmatrix}
\]

Multiply row 2 by -6 and add to row 3:

\[
\begin{bmatrix}
1 & 0 & 3/5 \\
0 & 1 & -6/5 \\
0 & 0 & 1/5
\end{bmatrix}
\begin{bmatrix}
-1/5 & 2/5 & 0 \\
2/5 & 1/5 & 0 \\
8/5 & -6/5 & 1
\end{bmatrix}
\]

Multiply row 3 by 5:

\[
\begin{bmatrix}
1 & 0 & 3/5 \\
0 & 1 & -6/5 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1/5 & 2/5 & 0 \\
2/5 & 1/5 & 0 \\
8 & -6 & 5
\end{bmatrix}
\]

Multiply row 3 by 6/5 and add to row 2:

\[
\begin{bmatrix}
1 & 0 & 3/5 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1/5 & 2/5 & 0 \\
10 & -7 & 6 \\
8 & -6 & 5
\end{bmatrix}
\]

Multiply row 3 by -3/5 and add to row 1:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-5 & 4 & -3 \\
10 & -7 & 6 \\
8 & -6 & 5
\end{bmatrix} = A^{-1}
\]
*15. First find $A^{-1}$ by the method of Exercise 14:

$$A = \begin{bmatrix} 1 & 1 \\ 24 & 14 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply row 1 by -24 and add to row 2:

$$\begin{bmatrix} 1 & 1 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -24 & 1 \end{bmatrix}$$

Multiply row 2 by -1/10:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 24/10 & -1/10 \end{bmatrix}$$

Multiply row 2 by -1 and add to row 1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -14/10 & 1/10 \\ 24/10 & -1/10 \end{bmatrix} = A^{-1}$$

Now multiply $A^{-1} \cdot B$:

$$\begin{bmatrix} -14/10 & 1/10 \\ 24/10 & -1/10 \end{bmatrix} \begin{bmatrix} 70 \\ 1180 \end{bmatrix} = \begin{bmatrix} 20 \\ 50 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

so the solution is $x = 20$, $y = 50$.

16. First find the inverse of the matrix of coefficients:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply row 1 by -1 and add to row 2:

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Multiply row 2 by -1:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$
Multiply row 2 by -2 and add to row 1:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 2 \\
1 & -1
\end{bmatrix}
\]

Now multiply \(A^{-1}\) \(B\):

\[
\begin{bmatrix}
-1 & 2 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
-4 \\
5
\end{bmatrix} = \begin{bmatrix}
14 \\
9
\end{bmatrix}
\]

so the solution is \(x = 14, y = -9\).

17. a. \(A^T = \begin{bmatrix}
1 & 6 \\
3 & -2 \\
4 & 1
\end{bmatrix}\)

b. If \(A\) is symmetric then \(a_{ij} = a_{ji}\) and \(A^T(i, j) = A(j, i) = A(i, j)\).

Therefore \(A^T = A\). If \(A^T = A\), then \(A(i, j) = A^T(i, j) = A(j, i)\) and \(A\) is symmetric.

c. \((A^T)^T = A\) follows from the definition - two interchanges of row and column gets back to the original.

d. Let \(A + B = C\). Then \(C^T(i, j) = C(j, i) = A(j, i) + B(j, i) = A^T(i, j) + B^T(i, j)\)

and \(C^T = A^T + B^T\)

e. Let \(A\) be an \(n \times m\) matrix and \(B\) be an \(m \times p\) matrix; then \(A^T\) is \(m \times n\) and \(B^T\) is \(p \times m\).

Let \(A \cdot B = C\). Then \(C^T(i, j) = C(j, i) = \sum_{k=1}^{m} a_{jk} b_{ki} = \sum_{k=1}^{m} A^T(k, j) B^T(i, k) = \sum_{k=1}^{m} B^T(i, k) A^T(k, j) = (B^T \cdot A^T)(i, j)\) and \(C^T = B^T \cdot A^T\).

*18. For example, \[
\begin{bmatrix}
1 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
-1 & -1
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}\]

19. For example, \[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix} = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} = \begin{bmatrix}
3 & 6 \\
3 & 6
\end{bmatrix}\]

but \[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix} \neq \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}\]

20. This is not always true (for example, use the \(A\) and \(B\) of Practice 45). It is true if \(A = B = I\), for example.
21. According to the Law of Cosines, in the following triangle

\[ a^2 = b^2 + c^2 - 2bc \cos \theta. \] Also, \( b = \|\mathbf{V}\| = \sqrt{v_1^2 + v_2^2}, \ c = \|\mathbf{U}\| = \sqrt{u_1^2 + u_2^2}, \) and, by the distance formula, \( a^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2. \) Therefore

\[ (u_1 - v_1)^2 + (u_2 - v_2)^2 = v_1^2 + v_2^2 + u_1^2 + u_2^2 - 2 \|\mathbf{V}\| \cdot \|\mathbf{U}\| \cos \theta \]

\[ \cos \theta = \frac{\mathbf{U} \cdot \mathbf{V}}{\|\mathbf{U}\| \cdot \|\mathbf{V}\|} \]

22.*a. The recurrence relation is in the form of Equation (1) of Section 2.5, where \( c = 7 \) and \( g(n) = 0; \) the solution is given by Equation (6) of Section 2.5 and is \( M(n) = 7^{\log n}. \)

b. \( A(1) = 0 \) because no additions are required to multiply two \( 1 \times 1 \) matrices.

To compute the product of two \( n \times n \) matrices requires \( 7 \) multiplications of \( (n/2 \times n/2) \) matrices, each of which requires \( \binom{n}{2} \) additions. The product also requires \( 18 \) additions of \( (n/2 \times n/2) \) matrices, each of which requires \( (n/2)^2 \) additions.

c. Again using Equation (6) of Section 2.5, the solution is

\[ A(n) = \sum_{i=1}^{\log n} 7^{\log n-i} \left( \frac{2^i}{2} \right)^2 = 7^{\log n} \frac{18}{4} \sum_{i=1}^{\log n} 7^{-i} 2^{2i} = 7^{\log n} \frac{9}{2} \sum_{i=1}^{\log n} \left( \frac{2}{7} \right)^i = 7^{\log n} \frac{9}{2} \sum_{i=1}^{\log n} \left( \frac{4}{7} \right)^i = \]

\[ 7^{\log n} \frac{9}{2} \cdot \frac{4}{7} \left[ 1 + \frac{4}{7} + \left( \frac{4}{7} \right)^2 + \cdots + \left( \frac{4}{7} \right)^{\log n-1} \right] \] (using formula for the sum of terms of a geometric sequence)

\[ 7^{\log n} \frac{9}{2} \cdot \frac{4}{7} \frac{7}{3} \left[ 1 - \left( \frac{4}{7} \right)^{\log n} \right] = \Theta(7^{\log n}) \]

d. \( \log n \log 7 = \log 7 \log n \)

e. The total work is \( \Theta(7^{\log n}) + \Theta(7^{\log n}) = \Theta(7^{\log n}) = \Theta(n^{\log 7}) \equiv \Theta(n^{2.8}), \) which is better than the \( \Theta(n^3) \) of the traditional algorithm.
*23. \( A \land B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad A \lor B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\)

\( A \times B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B \times A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\)

24. \( A \land B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A \lor B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\)

\( A \times B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B \times A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\)

25. \( A \land B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A \lor B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\)

\( A \times B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad B \times A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\)

26. In order for \( A \lor B = A \land B \), it must be the case that \( a_{ij} \lor b_{ij} = a_{ij} \land b_{ij} \) for all \( i, j \). This is true if \( a_{ij} = b_{ij} \) or \( a_{ij} = b_{ij} = 0 \), therefore when \( A = B \).

27. The \( i,j \) entry in \( A \lor B \) is \( \max(a_{ij}, b_{ij}) = \max(b_{ij}, a_{ij}) \) = the \( i,j \) entry in \( B \lor A \). A similar argument holds for \( A \land B \).

28. Because the matrix is symmetric, only the entries on and below the main diagonal need be known. These entries will determine the entries above the main diagonal.

\[
\begin{array}{cccccc}
& x & & & & \\
\times & x & x & & & \\
x & x & x & x & & \\
x & x & x & x & x & \\
\ldots & x & x & x & x & x \\
x & x & x & x & x & x
\end{array}
\]

There are \( 1 + 2 + 3 + \ldots + n = n(n + 1)/2 \) such entries. Each entry can be either 0 or 1. The number of possibilities is therefore

\[
\frac{n(n+1)}{2} \cdot 2^\frac{n(n+1)}{2}
\]
29. The \( i,j \) entry of \( A^2 \) is \( \sum_{k=1}^{n} a_{ik}a_{kj} \)

The \( j,i \) entry of \( A^2 \) is \( \sum_{k=1}^{n} a_{jk}a_{ki} \)

But these are the same because \( a_{ik} = a_{ki} \) and \( a_{kj} = a_{jk} \) (\( A \) is symmetric)

30. \((A\cdot A^T)^T = (A^T)^T \cdot A^T\) by Exercise 17(e)
   \[= A \cdot A^T\] by Exercise 17(c)

Therefore \( A \cdot A^T \) is its own transpose, so it is symmetric by Exercise 17(b).

31. For \( n = 1 \),
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F(2) & F(1) \\ F(1) & F(0) \end{bmatrix}
\]

Assume that \( A^k = \begin{bmatrix} F(k+1) & F(k) \\ F(k) & F(k-1) \end{bmatrix} \)

Then \( A^{k+1} = A^k \cdot A = \begin{bmatrix} F(k+1) & F(k) \\ F(k) & F(k-1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F(k+1) + F(k) & F(k+1) \\ F(k) + F(k-1) & F(k) \end{bmatrix} = \begin{bmatrix} F(k+2) & F(k+1) \\ F(k+1) & F(k) \end{bmatrix} \)