15. Discuss in what sense the equivalences of Theorem 2.13 (page 117) form the basis of an algorithm which, given \( \phi \), pushes quantifiers to the top of the formula’s parse tree. If the result is \( \psi \), what can you say about commonalities and differences between \( \phi \) and \( \psi \)?

Exercises 2.4

* 1. Consider the formula \( \phi \overset{\text{def}}{=} \forall x \forall y Q(g(x, y), g(y, y), z) \), where \( Q \) and \( g \) have arity 3 and 2, respectively. Find two models \( M \) and \( M’ \) with respective environments \( l \) and \( l’ \) such that \( M \models l \phi \) but \( M’ \not\models l’ \phi \).

2. Consider the sentence \( \phi \overset{\text{def}}{=} \forall x \exists y \exists z (P(x, y) \land P(z, y) \land (P(x, z) \rightarrow P(z, x))) \). Which of the following models satisfies \( \phi \)?
   (a) The model \( M \) consists of the set of natural numbers with \( P_M \overset{\text{def}}{=} \{(m, n) \mid m < n\} \).
   (b) The model \( M’ \) consists of the set of natural numbers with \( P_{M’} \overset{\text{def}}{=} \{(m, 2 * m) \mid m \text{ natural number}\} \).
   (c) The model \( M” \) consists of the set of natural numbers with \( P_{M”} \overset{\text{def}}{=} \{(m, n) \mid m < n + 1\} \).

3. Let \( P \) be a predicate with two arguments. Find a model which satisfies the sentence \( \forall x \neg P(x, x) \); also find one which doesn’t.

4. Consider the sentence \( \forall x (\exists y P(x, y) \land (\exists z P(z, x) \rightarrow \forall y P(x, y))) \). Please simulate the evaluation of this sentence in a model and look-up table of your choice, focusing on how the initial look-up table \( l \) grows and shrinks like a stack when you evaluate its subformulas according to the definition of the satisfaction relation.

5. Let \( \phi \) be the sentence \( \forall x \forall y \exists z (R(x, y) \rightarrow R(y, z)) \), where \( R \) is a predicate symbol of two arguments.
   * (a) Let \( A \overset{\text{def}}{=} \{a, b, c, d\} \) and \( R_M \overset{\text{def}}{=} \{(b, c), (b, b), (b, a)\} \). Do we have \( M \models \phi \)? Justify your answer, whatever it is.
   * (b) Let \( A’ \overset{\text{def}}{=} \{a, b, c\} \) and \( R_{M’} \overset{\text{def}}{=} \{(b, c), (a, b), (c, b)\} \). Do we have \( M’ \models \phi \)? Justify your answer, whatever it is.

* 6. Consider the three sentences
   \[
   \phi_1 \overset{\text{def}}{=} \forall x P(x, x) \\
   \phi_2 \overset{\text{def}}{=} \forall x \forall y (P(x, y) \rightarrow P(y, x)) \\
   \phi_3 \overset{\text{def}}{=} \forall x \forall y \forall z ((P(x, y) \land P(y, z) \rightarrow P(x, z)))
   \]
   which express that the binary predicate \( P \) is reflexive, symmetric and transitive, respectively. Show that none of these sentences is semantically entailed by the other ones by choosing for each pair of sentences above a model which satisfies these two, but not the third sentence – essentially, you are asked to find three binary relations, each satisfying just two of these properties.
7. Show the semantic entailment $\forall x \neg \phi \models \neg \exists x \phi$; for that you have to take any model which satisfies $\forall x \neg \phi$ and you have to reason why this model must also satisfy $\neg \exists x \phi$. You should do this in a similar way to the examples in Section 2.4.2.

8. Show the semantic entailment $\forall x P(x) \lor \forall x Q(x) \models \forall x (P(x) \lor Q(x))$.

9. Let $\phi$ and $\psi$ and $\eta$ be sentences of predicate logic.
   (a) If $\psi$ is semantically entailed by $\phi$, is it necessarily the case that $\psi$ is not semantically entailed by $\neg \phi$?

   * (b) If $\psi$ is semantically entailed by $\phi \land \eta$, is it necessarily the case that $\psi$ is semantically entailed by $\phi$ and semantically entailed by $\eta$?

   (c) If $\psi$ is semantically entailed by $\phi$ or by $\eta$, is it necessarily the case that $\psi$ is semantically entailed by $\phi \lor \eta$?

   (d) Explain why $\psi$ is semantically entailed by $\phi$ iff $\phi \rightarrow \psi$ is valid.

10. Is $\forall x (P(x) \lor Q(x)) \models \forall x P(x) \lor \forall x Q(x)$ a semantic entailment? Justify your answer.

11. For each set of formulas below show that they are consistent:
   (a) $\forall x \neg S(x, x)$, $\exists x P(x)$, $\forall x \exists y S(x, y)$, $\forall x (P(x) \rightarrow \exists y S(y, x))$

   * (b) $\forall x \neg S(x, x)$, $\forall x \exists y S(x, y)$,
       $\forall x \forall y \forall z ((S(x, y) \land S(y, z)) \rightarrow S(x, z))$

   (c) $\forall x (P(x) \lor Q(x)) \rightarrow \exists y R(y)$, $\forall x (R(x) \rightarrow Q(x))$, $\exists y (\neg Q(y) \land P(y))$

   * (d) $\exists x S(x, x)$, $\forall x \forall y (S(x, y) \rightarrow (x = y))$.

12. For each of the formulas of predicate logic below, either find a model which does not satisfy it, or prove it is valid:
   (a) $\forall x \forall y (S(x, y) \rightarrow S(y, x)) \rightarrow (\forall x \neg S(x, x))$

   * (b) $\exists y ((\forall x P(x)) \rightarrow P(y))$

   (c) $\forall x (P(x) \rightarrow \exists y Q(y))) \rightarrow (\forall x \exists y (P(x) \rightarrow Q(y)))$

   (d) $\forall x \exists y (P(x) \rightarrow Q(y))) \rightarrow (\forall x (P(x) \rightarrow \exists y Q(y)))$

   (e) $\forall x \forall y \forall z ((S(x, y) \land S(z, y)))$

   (f) $\forall x \forall y (S(x, y) \rightarrow (x = y)) \rightarrow (\forall z \neg S(z, z))$

   * (g) $\forall x \exists y (S(x, y) \land ((S(x, y) \land S(y, x)) \rightarrow (x = y)))$ $\rightarrow$
       $(\neg \exists z \forall w (S(z, w)))$.

   (h) $\forall x \forall y ((P(x) \rightarrow P(y)) \land (P(y) \rightarrow P(x)))$

   (i) $\forall x ((P(x) \rightarrow Q(x)) \land (Q(x) \rightarrow P(x))) \rightarrow ((\forall x P(x)) \rightarrow (\forall x Q(x)))$

   (j) $((\forall x P(x)) \rightarrow (\forall x Q(x)) \rightarrow (\forall x ((P(x) \rightarrow Q(x)) \land (Q(x) \rightarrow P(x))))$)

   (k) Difficult: $\forall x \exists y (P(x) \rightarrow Q(y))) \rightarrow (\exists y \forall x (P(x) \rightarrow Q(y)))$.

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**Exercises 2.5**

1. Assuming that our proof calculus for predicate logic is sound (see exercise 3 below), show that the validity of the following sequents cannot be proved by finding for each sequent a model such that all formulas to the left of $\vdash$ evaluate to $\top$ and the sole formula to the right of $\vdash$ evaluates to $\bot$ (explain why this guarantees the non-existence of a proof):