# PROBLEM SOLVING

NCTM's *Principles and Standards for School Mathematics* recommends that the mathematics curriculum "include numerous and varied experiences with problem solving as a method of inquiry and application." There are many problems within the MATHCOUNTS program that may be considered difficult if attacked by setting up a series of equations, but quite simple when attacked by problem-solving strategies such as looking for a pattern, drawing a diagram, making an organized list, and so on.

The problem-solving method that will be used in the following discussion consists of four basic steps:

| | |
|---|---|
| **FIND OUT** | Look at the problem.<br>Have you seen a similar problem before?<br>If so, how is this problem similar? How is it different?<br>What facts do you have?<br>What do you know that is not stated in the problem? |
| **CHOOSE A STRATEGY** | How have you solved similar problems in the past?<br>What strategies do you know?<br>Try a strategy that seems as if it will work.<br>If it doesn't, it may lead you to one that will. |
| **SOLVE IT** | Use the strategy you selected and work the problem. |
| **LOOK BACK** | Reread the question.<br>Did you answer the question asked?<br>Is your answer in the correct units?<br>Does your answer seem reasonable? |

Specific strategies may vary in name. Most, however, fall into these basic categories:

- Compute or Simplify (C)
- Use a Formula (F)
- Make a Model or Diagram (M)
- Make a Table, Chart or List (T)
- Guess, Check and Revise (G)
- Consider a Simpler Case (S)
- Eliminate (E)
- Look for Patterns (P)

To assist in using these problem-solving strategies, the answers to the Stretches, Warm-Ups and Workouts have been coded to indicate possible strategies. The single-letter codes above for each strategy appear in parentheses after each answer.

In the next section, the strategies above are applied to previously-published MATHCOUNTS problems. Examples of relevant problems from this year's School Handbook are also included for each section.

## Compute or Simplify (C)

Many problems are straightforward and require nothing more than the application of arithmetic rules. When solving problems, simply apply the rules and remember the order of operations.

> Given $(6^3)(5^4) = (N)(900)$, find $N$.

FIND OUT — What are we asked? The value of $N$ that satisfies an equation.

CHOOSE A STRATEGY — Will any particular strategy help here? Yes, factor each term in the equation into primes. Then, solve the equation noting common factors on both sides of the equation.

SOLVE IT — Break down the equation into each term's prime factors.

$$6^3 = 6 \times 6 \times 6 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$$

$$5^4 = 5 \times 5 \times 5 \times 5$$

$$900 = 2 \times 2 \times 3 \times 3 \times 5 \times 5$$

Two 2's and two 3's from the factorization of $6^3$ and two 5's from the factorization of $5^4$ cancel the factors of 900. The equation reduces to $2 \times 3 \times 5 \times 5 = N$, so $N = 150$.

LOOK BACK — Did we answer the question asked? Yes.
Does our answer make sense? Yes—since $900 = 30^2 = (2 \times 3 \times 5)^2$, we could have eliminated two powers of 2, 3 and 5 to obtain the same answer.

*Example* — Problem #6 of Warm-Up 12 is a good example of a problem that can be solved quickly if the idea of "difference of squares" is applied to first simplify the expression.

## Use a Formula (F)

Formulas are one of the most powerful mathematical tools at our disposal. Often, the solution to a problem involves substituting values into a formula or selecting the proper formula to use. Some of the formulas that will be useful for students to know are listed on page 31. However, other formulas will be useful to students, too. If the strategy code for a problem is (F), then the problem can be solved with a formula. When students encounter problems for which they don't know an appropriate formula, they should be encouraged to discover the formula for themselves.

> The formula F = 1.8C + 32 can be used to convert temperatures between degrees Fahrenheit (F) and degrees Celsius (C). How many degrees are in the Celsius equivalent of –22°F?

FIND OUT — What are we trying to find? We want to know a temperature in degrees Celsius instead of degrees Fahrenheit.

CHOOSE A STRATEGY — Since we have a formula which relates Celsius and Fahrenheit temperatures, let's replace F in the formula with the value given for degrees Fahrenheit.

SOLVE IT — The formula we're given is F = 1.8C + 32. Substituting –22 for F in the equation leads to the following solution:

$$\begin{aligned} -22 &= 1.8C + 32 \\ -22 - 32 &= 1.8C \\ -30 &= C \end{aligned}$$

The answer is –30°C.

LOOK BACK — Is our answer reasonable? Yes.

*Example* — The formula for determining the area of a rhombus is used to answer problem #6 in Workout 4, and is discussed in detail in the Answer Key.

# Make a Model (M)

Mathematics is a way of modeling the real world. A mathematical model has traditionally been a form of an equation. The use of physical models is often useful in solving problems. There may be several models appropriate for a given problem. The choice of a particular model is often related to the student's previous knowledge and problem-solving experience. Objects and drawings can help to visualize problem situations. Acting out is also a way to visualize the problem. Writing an equation is an abstract way of modeling a problem situation. The use of modeling provides a method for organizing information that could lead to the selection of another problem-solving strategy.

## Use Physical Models

> Four holes are drilled in a straight line in a rectangular steel plate. The distance between hole 1 and hole 4 is 35mm. The distance between hole 2 and hole 3 is twice the distance between hole 1 and hole 2. The distance between hole 3 and hole 4 is the same as the distance between hole 2 and hole 3. What is the distance, in millimeters, between the center of hole 1 and the center of hole 3?

FIND OUT — We want to know the distance between hole 1 and hole 3.
What is the distance from hole 1 to hole 4? 35 mm.
What is the distance from hole 1 to hole 2? Half the distance from hole 2 to hole 3.
What is the distance from hole 3 to hole 4? The same as from hole 2 to hole 3.

CHOOSE A STRATEGY — Make a model of the problem to determine the distances involved.

SOLVE IT — Mark off a distance of 35 mm.
Place a marker labeled #1 at the zero point and one labeled #4 at the 35-mm point.
Place markers #2 and #3 between #1 and #4.
1) Move #2 and #3 until the distances between #2 & #3 and #3 & #4 are equal.
2) Is the distance between #1 & #2 equal to half the distance between #2 & #3?
Adjust the markers until both of these conditions are met.
Measure the distances to double check. The distance between #1 and #3 is 21 mm.

LOOK BACK — Does our answer seem reasonable? Yes, the answer must be less than 35.

*Example* — Allowing students to experiment with small unit cubes is a great way to encourage exploration of the possibilities for problem #5 in Workout 9.

## Act Out the Problem

There may be times when you experience difficulty in visualizing a problem or the procedure necessary for its solution. In such cases you may find it helpful to physically act out the problem situation. You might use people or objects exactly as described in the problem, or you might use items that represent the people or objects. Acting out the problem may itself lead you to the answer, or it may lead you to find another strategy that will help you find the answer. Acting out the problem is a strategy that is very effective for young children.

> There are five people in a room and each person shakes every other person's hand exactly one time. How many handshakes will there be?

FIND OUT — We are asked to determine the total number of handshakes.
How many people are there? Five.
How many times does each person shake another's hand? Only once.

CHOOSE A STRATEGY — Would it be possible to model this situation in some way? Yes, pick five friends and ask them to act out the problem.

Should we do anything else? Keep track of the handshakes with a list.

SOLVE IT — Get five friends to help with this problem.
Make a list with each person's name at the top of a column.
Have the first person shake everyone's hand. How many handshakes were there? Four.
Repeat this four more times with the rest of the friends. Write down who each person shook hands with. Our table should look something like this:

| Rhonda | Jagraj | Rosario | Kiran | Margot |
|---|---|---|---|---|
| Jagraj | Rosario | Kiran | Margot | Rhonda |
| Rosario | Kiran | Margot | Rhonda | Jagraj |
| Kiran | Margot | Rhonda | Jagraj | Rosario |
| Margot | Rhonda | Jagraj | Rosario | Kiran |

There were a total of twenty handshakes. But notice that each person actually shook everyone else's hand twice. (For example, Rhonda shook Jagraj's hand, and Jagraj shook Rhonda's hand.) Divide the total number of handshakes by two to find out the total number if each person had shaken every other person's hand only once. There were ten handshakes.

LOOK BACK — Did we answer the question? Yes.
Does our answer seem reasonable? Yes.

*Example* — Acting out the scenario for problem #4 in Warm-Up 1 will quickly lead students to the very unexpected answer!

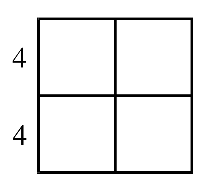
## Use Drawings or Sketches

> If an eight-inch square cake serves four people, how many twelve-inch square cakes are needed to provide equivalent servings to eighteen people?

FIND OUT — We are to find how many $12 \times 12$ cakes are needed.
How big is the original cake? $8 \times 8$.
How many people did it feed? 4.
How big are the other cakes? $12 \times 12$.
How many people must they feed? 18.

CHOOSE A STRATEGY — How should we approach this problem? Diagram the cakes to understand the size of the portions.

SOLVE IT — Draw an $8 \times 8$ cake and cut it into 4 equal pieces. Since each piece is a square with side length of 4, the area of each piece is $4 \times 4 = 16$ square inches.




So each person gets 16 square inches of cake.

18 people times 16 square inches per person equals 288 total square inches of cake needed.

We know that a $12 \times 12$ cake contains 144 square inches of cake.

288 divided by 144 equals 2, so two $12 \times 12$ cakes are required to feed 18 people.

LOOK BACK Did we answer the correct question, and does our answer seem reasonable? Yes.

*Example* Drawing a picture is really the only way to see the total area that is involved in Warm-Up 16, problem #3.

### Use Equations

> Lindsey has a total of $82.00, consisting of an equal number of pennies, nickels, dimes and quarters. How many coins does she have in all?

FIND OUT We want to know how many coins Lindsey has.
How much money does she have total? $82.00.
How many of each coin does she have? We don't know exactly, but we know that she has an equal number of each coin.

CHOOSE A STRATEGY We know how much each coin is worth, and we know how much all of her coins are worth total, so we can write an equation that models the situation.

SOLVE IT Let $p$ be the number of pennies, $n$ the number of nickels, $d$ the number of dimes, and $q$ the number of quarters.
We then have the equation $p + 5n + 10d + 25q = 8200$.
We know that she has an equal number of each coin, so $p = n = d = q$. Substituting $p$ for the other variables gives an equation in just one variable. The equation above becomes $p + 5p + 10p + 25p = 41p = 8200$, so $p = 200$.
Lindsey has 200 pennies. Since she has an equal number of each coin, she also has 200 nickels, 200 dimes and 200 quarters. Therefore, she has 800 coins.

LOOK BACK Did we answer the question asked? Yes.
Does our answer seem reasonable? Yes, we know the answer must be less than 8200 (the number of coins if they were all pennies) and greater than 328 (the number of coins if they were all quarters).

*Example* Though not completely necessary, a system of equations can be used to calculate the price of a pencil for problem #6 in Warm-Up 1.

## Make a Table, Chart or List (T)

Making a table, chart, graph, or list is a way to organize data presented in a problem. This problem-solving strategy allows the problem solver to discover relationships and patterns among data.

### Use Tree Diagrams or Organized Lists

> Customers at a particular yogurt shop may select one of three flavors of yogurt. They may choose one of four toppings. How many one-flavor, one-topping combinations are possible?

FIND OUT What question do we have to answer? How many flavor-topping combinations are possible?
How many flavors are available? Three.
How many toppings are available? Four.

Are you allowed to have more than one flavor or topping? No, the combinations must have only one flavor and one topping.

CHOOSE A STRATEGY — How could we organize the possible combinations to help? With letters and numbers in a list.

SOLVE IT — Make an organized list. Use F and T to denote either flavor or topping. Use the numbers 1 – 3 and 1 – 4 to mark different flavors and toppings.

F1T1, F1T2, F1T3, F1T4
F2T1, F2T2, F2T3, F2T4
F3T1, F3T2, F3T3, F3T4

Now count the number of combinations. There are 12 combinations possible.

LOOK BACK — Did we answer the question asked? Yes.
Does our answer seem reasonable? Yes.

*Example* — As shown in the Representation for problem #5 in Warm-Up 11, an organized list is helpful in determining the value of *a*, *b*, *c* and *d*.

## Make a Chart

How many hours will a car travelling at 45 miles per hour take to catch up with a car travelling at 30 miles per hour if the slower car starts one hour before the faster car?

FIND OUT — What is the question we have to answer? How long does it take for the faster car to catch the slower car.
What is the speed of the slower car? 30 miles per hour.
What is the speed of the faster car? 45 miles per hour.

CHOOSE A STRATEGY — What strategy will help here? We could model this on paper, but accuracy would suffer. We could also use equations. But let's make a table with the time and distance traveled since that will explicitly show what's happening here.

SOLVE IT — Make a table with two rows and four columns.
The rows will identify the cars, and the columns will mark the hours.
Where the rows and columns intersect will indicate distance traveled, since distance equals the speed times the amount of time traveled.

| Car \ Hour | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Slow Car | 30 | 60 | 90 | 120 |
| Fast Car | 0 | 45 | 90 | 135 |

At the end of the first hour, the faster car was just starting. At the end of the second hour, the faster car had gone 45 miles. At the end of the third hour, the faster car had gone 90 miles. This equals the distance traveled by the slower car in three hours. So, the faster car only traveled for two hours.

LOOK BACK — Did we answer the question asked? Yes.
Does our answer seem reasonable? Yes.

*Example* — A chart is used in the Representation for Workout 3, problem #3.

# Guess, Check and Revise (G)

The guess-check-and-revise strategy for problem solving can be helpful for many types of problems. When using this strategy, students are encouraged to make a reasonable guess, check the guess, and revise the guess if necessary. By repeating this process a student can arrive at a correct answer that has been checked. Using this strategy does not always yield a correct solution immediately but provides information that can be used to better understand the problem and may suggest the use of another strategy. Students have a natural affinity for this strategy and should be encouraged to use it when appropriate.

To use the guess-check-and-revise strategy, follow these steps:

1. Make a guess at the answer.
2. Check your guess. Does it satisfy the problem?
3. Use the information obtained in checking to help you make a new guess.
4. Continue the procedure until you get the correct answer.

> Leah has $4.05 in dimes and quarters. If she has 5 more quarters than dimes, how many of each does she have?

FIND OUT — What are we asked to determine? We need to find how many dimes and how many quarters Leah has.
What is the total amount of money? $4.05.
What else do we know? There are five more quarters than dimes.

CHOOSE A STRATEGY — Will listing combinations help? Yes, but creating an extended list of possible combinations of dimes and quarters could be cumbersome to create.
What other strategy would work? Pick a number, try it, and adjust the estimate.

SOLVE IT — Try 5 dimes. That would mean 10 quarters.
$5 \times \$0.10 + 10 \times \$0.25 = \$3.00$
Increase the number of dimes to 7.
$7 \times \$0.10 + 12 \times \$0.25 = \$3.70$
Try again. This time use 8 dimes.
$8 \times \$0.10 + 13 \times \$0.25 = \$4.05$
Leah has 8 dimes and 13 quarters.

LOOK BACK — Did we answer the question asked, and does our answer seem reasonable? Yes.

> Trevor had 60 markers he could turn in at the end of the year for extra-credit points he had earned during the year. Some markers were worth one point and others were worth two points. If he was entitled to a total of 83 extra-credit points, how many one point markers did he have?

FIND OUT — What question are we trying to answer? The question is how many one-point markers did Trevor have.
What is the total number of markers he had? 60.
What were their possible values? One or two points.
What was the total value of all the markers? The markers totaled 83 points.

CHOOSE A STRATEGY — How can we approach this problem? Make a table of the possible number of markers and their total value.

SOLVE IT — Make a guess as to the first value. We can adjust our guess as we get closer to the desired answer.
Pick 10 as the number of one-point markers. This means he has 50 two-point markers since we know he has 60 markers total. The value of this combination is 110 points.
We can keep track of our guesses in a table by listing the number of one-point markers, the number of two-point markers, and the total number of points various combinations would give.

| # of 1-point Markers | # of 2-point Markers | Total Value |
|---|---|---|
| 10 | 50 | 110 |
| 50 | 10 | 70 |
| 40 | 20 | 80 |
| 38 | 22 | 82 |
| 37 | 23 | 83 |

Trevor had 37 one-point markers.

LOOK BACK Did we answer the question? Yes.
Does our answer seem reasonable? Yes, we know the answer has to be less than 60.
Also, 23 points more than 60 implies that 23 markers were worth 2 points.

*Example* By logically guessing, checking and revising, it would not take too long to determine the number of regular packs of batteries that were sent in problem #1 of Warm-Up 6.

## Consider a Simpler Case (S)

The problem-solving strategy of simplifying is most often used in conjunction with other strategies. Writing a simpler problem is one way of simplifying the problem-solving process. Rewording the problem, using smaller numbers, or using a more familiar problem setting may lead to an understanding of the solution strategy to be used. Many problems may be divided into simpler problems to be combined to yield a solution. Some problems can be made simpler by working backwards.

Sometimes a problem is too complex to solve in one step. When this happens, it is often useful to simplify the problem by dividing it into cases and solving each one separately.

### Divide into smaller problems

Three shapes—a circle, a rectangle, and a square—have the same area. Which shape has the smallest perimeter?

FIND OUT We want to know which of three shapes has the smallest perimeter.

CHOOSE A STRATEGY Will any particular strategy help here? Yes, we can compare the perimeters of the shapes pairwise. This will be easier than calculating the area of each since numbers are not given.

SOLVE IT First, compare the circumference of the circle to the perimeter of the square. They have equal area, so the area of the circle, $\pi r^2$, equals the area of the square, $s^2$. Consequently, the perimeter of the square will be slightly greater than the circumference of the circle.

Next, compare the perimeter of the square to the perimeter of the rectangle. A square is the quadrilateral which has minimum perimeter, so the perimeter of the square must be less than the perimeter of the rectangle.

By the transitive property, then, the perimeter of the rectangle will be greater than the circumference of the circle. Hence, the circle has the smallest perimeter.

LOOK BACK Did we answer the question asked? Yes.
Does our answer make sense? Yes. If we arbitrarily choose 100 units$^2$ as the area of each shape, the circumference of the circle is roughly 35.5 units, the perimeter of the square is 40 units, and the perimeter of the rectangle could be any amount greater than 40 units and less than 100 units.

*Example* Workout 1, problem #4 has many different parts to consider before the final answer can be determined. The amount charged by each of the cab companies must be calculated, but only after the length of time of the cab ride has been figured out. This problem really involves at least three different calculations.

## Work Backwards

> A student needs at least a 95% average to receive a grade of A. On the first three tests the student averaged 92%. What is the minimum a student must average on the last two tests to receive a grade of A?

FIND OUT — We are asked to find what a student must average on her last two tests to get an A.
What average is required for an A? 95%.
How many tests will be figured into the average? Five.
How many test has she taken so far? Three.
What is her average on the first three tests? 92%.

CHOOSE A STRATEGY — What strategy would work well in this situation? Work backwards from the minimum required average needed for an A to find the scores needed on the last two tests.

SOLVE IT — Work backwards from the required average on all five tests.
The average of the tests must be 95%. There are five tests so the total number of points scored on the five tests must be, at least, $5 \times 95 = 475$.
So far, the average is 92% on three tests. While we don't know all of the individual scores, the total number of points scored on the three tests must be $3 \times 92 = 276$.
475 points required minus 276 scored so far equals 199 required on the next two tests.
199 divided by 2 equals 99.5.
The student must average 99.5% on her next two tests if she is to get an A.

LOOK BACK — Did we answer the question asked? Yes.
Does our answer seem reasonable? Yes, we knew we were looking for a number between 95 and 100.

*Example* — In problem #10 of Warm-Up 12 we need to use some logic skills and some Guess & Check, but the primary reasoning requires us to work backwards through the problem to determine the initial number of fish.

# Eliminate (E)

The strategy of elimination is commonly used by people in everyday life. In a problem-solving context, students must list and then eliminate possible solutions based upon information presented in the problem. The act of selecting a problem-solving strategy is an example of the elimination process. Logical reasoning is a problem-solving strategy that is used in all problem-solving situations. It can result in the elimination of incorrect answers, particularly in "if-then" situations and in problems with a listable number of possible solutions.

> What is the largest two-digit number that is divisible by 3 whose digits differ by 2?

FIND OUT — What are we asked to find? A certain number.
What do we know about the number? The number is less than 100. It is divisible by 3.
The digits of the number differ by 2.

CHOOSE A STRATEGY — What strategy will help here? Working backwards from 99, list numbers and eliminate those that do not satisfy the conditions given. (Notice that we have already eliminated numbers greater than 99.)

SOLVE IT — 99, 98, 97, 96, 95, 94, 93, 92, 91, 90,
89, 88, 87, 86, 85, 84, 83, 82, 81, 80,
79, 78, 77, 76, 75, 74, 73, 72, 71, 70, . . .

Eliminate those numbers that are not divisible by 3:

**99**, ~~98~~, ~~97~~, **96**, ~~95~~, ~~94~~, **93**, ~~92~~, ~~91~~, **90**,
~~89~~, ~~88~~, **87**, ~~86~~, ~~85~~, **84**, ~~83~~, ~~82~~, **81**, ~~80~~,
~~79~~, **78**, ~~77~~, ~~76~~, **75**, ~~74~~, ~~73~~, **72**, ~~71~~, ~~70~~, . . .

From these, eliminate all numbers whose digits do not differ by 2:

~~99~~, ~~96~~, ~~93~~, ~~90~~, ~~87~~, ~~84~~, ~~81~~, ~~78~~, **75**, ~~72~~, . . .

75 is the largest number that remains.

LOOK BACK Did we answer the question asked? Yes.
Do we have a two-digit number divisible by 3 whose digits differ by 2? Yes.

*Example* Using the clues in problem #2 from Warm-Up 4, we can eliminate the values for *n* that don't meet the criteria until we are left with just one value for *n*.

## Look for Patterns (P)

When students use this problem-solving strategy, they are required to analyze patterns in data and make predictions and generalizations based on their analysis. They then must check the generalization against the information in the problem and possibly make a prediction from, or extension of, the given information. A pattern is a regular, systematic repetition. A pattern may be numerical, visual or behavioral. By identifying the pattern, you can predict what will come next and what will happen again and again in the same way. Looking for patterns is a very important strategy for problem solving and is used to solve many different kinds of problems. Sometimes you can solve a problem just by recognizing a pattern, but often you will have to extend a pattern to find a solution. Making a number table often reveals patterns, and for this reason it is frequently used in conjunction with this strategy.

> Laura was given an ant farm by her grandparents for her 13th birthday. The farm could hold a total of 100,000 ants. Laura's farm had 1500 ants when it was given to her. If the number of ants in the farm on the day after her birthday was 3000 and the number of ants the day after that was 6000, in how many days will the farm be full?

FIND OUT We need to know when the ant farm will be full.
How many ants will the farm hold? 100,000.
How many ants are in the farm the first day? 1500.
How many ants are in the farm the second day? 3000.
How many ants are in the farm the third day? 6000.

CHOOSE A STRATEGY Is a pattern developing? Yes, each day twice as many ants are in the farm as the day before. Make a table to count the ants systematically.

SOLVE IT Draw a table with two lines for numbers.
The top line is the number of days after Laura's birthday, and the bottom line is the number of ants in the farm on that day.

| # days | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|---|
| # ants | 1500 | 3000 | 6000 | 12,000 | 24,000 | 48,000 | 96,000 | 192,000 |

The ant farm will be full seven days after her birthday.

LOOK BACK Read the question again. Did we answer all of the question? Yes.
Does our answer seem reasonable? Yes.
What assumption are we making? We are assuming that the pattern—the number of ants doubles each day—continues indefinitely.

*Example* In problem #6 from Workout 6, students may recognize that the pattern formed by the number of handshakes for increasing numbers of gymnasts is related to the triangular numbers. This is a great hint when attempting to solve the problem.