G. Polya and “How to Solve It!”

An overall framework for problem solving was described by G. Polya in a book called “How to Solve It!” (see supplementary references). Although Polya’s focus was on solving math problems, the strategies are much more general and are broadly applicable. Inductive reasoning is the basis of most of the creative processes in the “real world”. Physics in general and mechanics in particular provides an ideal laboratory for building skill in inductive reasoning and discovery.

Here is an outline of Polya’s framework:

1. **Understand the Problem** [*Identify the goal*]
   The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

   - What are the unknowns?
   - What are the given quantities?
   - What are the given conditions?
   - Are there any constraints?

   For many problems it is useful to

   - draw a diagram

   and identify the given and required quantities on the diagram. Usually it is necessary to

   - introduce suitable notation

   In choosing symbols for the unknown quantities we often use letters such as a, b, c, x, and y, but in most cases it helps to use initials as suggestive symbols, for instance, V for volume or t for time.

2. **Devise a Plan**
   Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps you to ask yourself explicitly: ”How can I relate the given to the unknown?” If you do not see a connection immediately, the following ideas may be helpful in devising a plan.

   - *Establish subgoals (divide into subproblems)*
     In a complex problem it is often useful to set subgoals. If we can first reach these subgoals, then we may be able to build on them to reach our final goal.
   - *Try to recognize something familiar*
     Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown or involves similar principles.
   - *Try to recognize patterns*
     Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it. [This is one reason you need to do lots of problems, so that you develop a base of patterns!]
   - *Use analogy*
     Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if the problem is in three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case. [One must do many problems to build a database of analogies!]

- *Introduce something extra*
  It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auziliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.
- *Take cases*
  We may sometimes have to split a problem into several cases and give a different solution for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.
- *Work backward (assume the answer)*
  It is often useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x-5=7$, we suppose that x is a number that satisfies $3x-5=7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x=4$. Since each of these steps can be reversed, we have solved the problem.
- *Indirect reasoning*
  Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q we assume that P is true and Q is false and try to see why this cannot happen.

3. **Carry out the Plan**
   In step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct. A string of equations is not enough!

4. **Look Back**
   Be critical of your result; look for flaws in your solutions (e.g., inconsistencies or ambiguities or incorrect steps). Be your own toughest critic! Can you check the result? Checklist of checks:

   - Is there an alternate method that can yield at least a partial answer?
   - Try the same approach for some similar but simpler problem.
   - Check units (*always, always, always!*).
   - If there is a numerical answer, is the order of magnitude correct or reasonable?
   - Trends. Does the answer vary as you expect if you vary one or more parameters? For example, if gravity is involved, does the answer change as expected if you vary g?
   - Check limiting cases where the answer is easy or known. Take the limit as variables or parameters reach certain values. For example, take a mass to be zero or infinite.
   - Check special cases where the answer is easy or known. This might be a special angle (0 or 45 or 90 degrees) or the case when all masses are set equal to each other.
   - Use symmetry. Does your answer reflect any symmetries of the physical situation?
   - If possible, do a simple experiment to see if your answer makes sense.

We will examine potential strategies as we solve problems. The emphasis here is on being *conscious* of our problem-solving strategies and on constructing a solution that reflects the steps outlined above.